ČECH COHOMOLOGY ON MANIFOLDS

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Abstract. In this project, we explore the theory of Čech cohomology in the setting of its application to smooth manifolds. We recall the definition of a smooth manifold, then define Čech cohomology and develop some basic theory. We then compare this cohomology theory to de Rham cohomology, which is defined using differential forms on a smooth manifold. In particular, we will give an exposition of Weil's proof of the somewhat surprising result that, despite their very different definitions, for all spaces, the two cohomology theories coincide. We then try to understand this coincidence more intuitively, by calculating the two cohomologies of a couple simple spaces.

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Preface

The following is an expository article/summary written for a summer project undertaken during the summer of 2025 under the supervision of Prof. Jason D. Lotay.

When writing this article, I had in mind an audience who has taken an undergraduate course in elementary point-set topology and is happy to black box some elementary category theory language and some elementary theory regarding differential forms; some extra preliminary material is briefly covered in the first section. I make no particular claim to originality, but would like to mention that my exposition of the construction of the Čech cohomology of a topological space has a larger focus on good covers and their nerves, using triangulations of spaces as the main source of examples and intuition. This results in a longer exposition of the construction than is strictly necessary or standard (compared to, for example, in [1] or [2]) but I hope this will make the exposition more digestible.

Much of my time spent during the project was on understanding the de Rham theory on manifolds, which included a brief encounter with fibre bundles and their characteristic classes (namely, the Euler class and the Thom class of an oriented vector bundle) and how these relate to Poincaré duals of closed submanifolds of an oriented manifolds; I have chosen to omit an exposition of these aside from the absolute bare essentials.

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Introduction

For a topological space M, the Čech cohomology of M is defined using purely topological data about M: it is defined by considering all open covers $\mathfrak{U}$ of M and the finite intersections of open sets in each $\mathfrak{U}$. We see how to construct the Čech cohomology of a topological space in Section 2. On the other hand, the de Rham cohomology of a smooth manifold M is defined using the differentiable structure on M which allow us to define differentiable forms on M; the theory of differential forms can be thought of as a generalisation of vector calculus in $\mathbb{R}^3$. A brief summary of the de Rham theory is given in Section 3.

Despite their very different definitions, (when taking coefficients in $\mathbb{R}$) the cohomology theories turn out to be isomorphic on smooth manifolds, a result sometimes called the *Čech-de Rham isomorphism*. We will explore a proof of this from the point of view of André Weil's Čech-de Rham Theory; Section 4 may be treated as an English exposition of Weil's original proof presented in [3] (subject to some notational changes for personal preference). A couple calculations are then carried out to get a better feeling for how this isomorphism works.

The main references of this expository piece are André Weil's paper [3] and the book by Raoul Bott and Loring W. Tu [1].

1. Conventions and Preliminaries

The primary purpose of this section is to introduce the conventions and language we will use throughout the exposition. We also take this as an opportunity to provide a brief exposition of some preliminary content.

1.1. Differential Complexes.

Definition 1.1.1. (Differential Complexes and Cohomology). A *differential complex* (of vector spaces) is a direct sum $C^* := \bigoplus_{n\in\mathbb{Z}} C^n$ of vector spaces indexed over $\mathbb{Z}$ together with a linear map $d : C^* \to C^*$, called the *differential* of C^*, such that $d^2 = d \circ d = 0$ and such that d restricts to linear maps $d : C^n \to C^{n+1}$ for all $n \in \mathbb{Z}$.

The vector space of n*-cochains* of C^* is C^n; the vector space of n*-cocycles* of C^* is $(\operatorname{Ker} d) \cap C^n = \operatorname{Ker}(d : C^n \to C^{n+1})$; and the vector space of n*-coboundaries* of C^* is $(\operatorname{Im} d \cap C^n) = \operatorname{Im}(d : C^{n-1} \to C^n)$. Because $d \circ d = 0$, we have for all n

$$0 \subseteq (\operatorname{Im} d) \cap C^n \subseteq (\operatorname{Ker} d) \cap C^n \subseteq C^n.$$

The *cohomology* of C^* is the direct sum $H^*(C^*) := \bigoplus_{n\in\mathbb{Z}} H^n(C^*)$, where $H^n(C^*)$ is the subquotient

$$H^n(C^*) := \frac{(\operatorname{Ker} d) \cap C^n}{(\operatorname{Im} d) \cap C^n}$$

and is called the n^{th} *cohomology* (vector space) of C^*. We sometimes refer to $H^n(C^*)$ as the cohomology of C^* in dimension n. If c is an n-cocycle, the *cohomology class* of c, sometimes denoted $[c]$, is the equivalence class of c in $H^n(C^*)$.

Remark 1.1.2. n-cocycles are sometimes called *closed* n-cochains. Similarly, n-coboundaries are sometimes called *exact* n-cochains.

Remark 1.1.3. What we call a differential complex (as defined in Definition 1.1.1) is sometimes called a *cochain complex of vector spaces* in other texts.

Definition 1.1.4. (Translations of Chain complexes). A useful operation one can do on a chain complex is shifting the indices by a fixed number, say $m \in \mathbb{Z}$. Given a chain complex C^* with differential d, the *translation* of C^* by m is the chain complex $C^{*+m} = \oplus_{n\in\mathbb{Z}} C^{n+m}$ with the same differential d. That is, n-cochains of C^{*+m} are $n+m$-cochains of C^*.

Definition 1.1.5. (Chain Maps). Let A^*, B^* be two differential complexes with differentials d_A, d_B, respectively. A linear map $f : A^* \to B^*$ is a *chain map* if

1. f commutes with differentials: $d_B \circ f = f \circ d_A$
2. for each $n \in \mathbb{Z}$, f restricts to a map $f : A^n \to B^n$

Remark 1.1.6. The definition of a chain map $f : A^* \to B^*$ is set up so that coboundaries are sent to coboundaries and cocycles are sent to cocycles; hence, chain maps induce maps $\overline{f} : H^*(A^*) \to H^*(B^*)$ on cohomology.

Definition 1.1.7. (Quasi-isomorphisms). Let A^*, B^* be two differential complexes. A chain map $f : A^* \to B^*$ is called a *quasi-isomorphism* if f induces an isomorphism $H^n(A^*) \xrightarrow{\sim} H^n(B^*)$ on cohomology for each n.

It is often the case that we have two chain maps $f, g : A^* \to B^*$ which are different on the nose, but induce the same maps on cohomology. One way to check when this will happen is to exhibit a *homotopy operator* between the two maps:

Definition 1.1.8. (Homotopy Operators). Let A^*, B^* be two differential complexes with differentials d_A, d_B respectively. Let $f, g : A^* \to B^*$ be two chain maps. A *homotopy operator* between f and g is a map $K : A^* \to B^{*-1}$ such that

$$f - g = Kd_A + d_B K.$$

Example 1.1.9. If there is a homotopy operator K between the identity map id_{A^*} on a differential complex A^* and the zero map, then the isomorphism on cohomology induced by the identity map is the zero map, thus the cohomology $H^*(A^*)$ of A^* is trivial; explicitly, if $a \in A^*$ is a cocycle, then $d_A a = 0$ and we see that $a = d_A K a$ is a coboundary.

1.2. **Manifolds.**

Definition 1.2.1. (Smooth Manifolds). *An m-dimensional smooth manifold* is a topological space M together with a set of ordered pairs $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ such that

1. M is Hausdorff
2. M is second-countable (i.e. the topology of M admits a countable basis)
3. $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ is an open cover of M
4. For $\alpha \in A$, $\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha$ is a homeomorphism of U_α onto an open subset V_α of $\mathbb{R}^m$
5. The transition functions $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are C^∞-maps.

The pairs $(U_\alpha, \varphi_\alpha)$ are called (coordinate) *charts*, and φ_α is called the *trivialisation*, or *local parametrisation*, of U_α. The open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ is sometimes called a *trivialising cover* of M. The datum $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is called an *atlas* (of M). A *maximal atlas* (also called a *differentiable structure*) on M is an atlas which is maximal with respect to inclusion. Each atlas of M is contained in a *unique* maximal atlas; that is, an atlas on M gives a *unique* differentiable structure on M.

Remark 1.2.2. The definition of an m-dimensional C^k-manifold is almost exactly the same as Definition 1.2.1: the only difference is we require the transition functions to be C^k-maps rather than C^∞-maps. Aside from in Theorem 2.2.13, **for us, all manifolds will be smooth** and so we will often omit the adjective "smooth."

Remark 1.2.3. We note that, whilst our definition is what will be more commonly found in the literature of the modern era, the definition of a manifold in [1] (or even in older texts such as [4]) requires the trivialisations in Definition 1.2.1 be onto $\mathbb{R}^m$ itself, rather than any open subset $V_\alpha \subseteq \mathbb{R}^m$. The advantage of our definition is that, in a maximal atlas, anything which could be a coordinate chart *is* a coordinate chart; the advantage of the convention in [1] is that locally the situation is exactly that of $\mathbb{R}^n$. In any case, given a manifold M and an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ in the sense of Definition 1.2.1, we can always *refine* the trivialising cover (see Definition 1.2.6) to get an atlas in the sense of [1]:

Firstly, recall that the topology of $\mathbb{R}^m$ is generated by open balls of $\mathbb{R}^m$[1]. Thus for every point $x \in U_\alpha$, there is an open neighbourhood $W_{\alpha,x}$ of x contained in U_α such that the restriction $\psi_{\alpha,x} := \varphi_\alpha |_{W_{\alpha,x}}$ of φ_α to $W_{\alpha,x}$ is a homeomorphism of $W_{\alpha,x}$ onto an open ball of $\mathbb{R}^m$. Letting $B = \{(\alpha, x) \mid \alpha \in A, x \in U_\alpha\}$, we see that $\{(W_\beta, \psi_\beta)\}_{\beta\in B}$ is an atlas of a manifold where the trivialisations are homeomor-

[1]The open balls here are, of course, with respect to the standard Euclidean metric on $\mathbb{R}^m$.

phisms onto open balls of $\mathbb{R}^m$. Since open balls of $\mathbb{R}^m$ are diffeomorphic to $\mathbb{R}^m$, there are diffeomorphisms $\phi_\beta : \psi_\beta(W_\beta) \xrightarrow{\sim} \mathbb{R}^m$ and we then see that $\{(W_\beta, \phi_\beta \circ \psi_\beta)\}_{\beta \in B}$ is an atlas of M in the sense of [1].

We also note that the differentiable structure given by $\{(W_\beta, \phi_\beta \circ \psi_\beta)\}_{\beta \in B}$ is the same as the one given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ since the charts in the former are compatible with the latter (see [5] §5.3 for details).

Definition 1.2.4. (Smooth Maps). Let N and M be manifolds of dimensions n and m with atlases $\{(U_\alpha, \varphi_\alpha)\}, \{(V_\beta, \psi_\beta)\}$, respectively. We say that a continuous map of manifolds $f : N \to M$ is smooth at a point $p \in N$ if for for all charts $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$ and all charts (V_β, ψ_β) with $f(p) \in V_\beta$, the map $\varphi_\beta \circ f \circ \varphi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth. We say $f : N \to M$ is smooth if it is smooth at every $p \in N$.

Remark 1.2.5. Smooth maps are the "canonical" type of maps between manifolds: if we have two manifolds M and N and a map between them $f : N \to M$, we almost always want f to be smooth so that we may use the differentiable structures on N and M.

Definition 1.2.6. (Refinements of Open Covers). Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathfrak{V} = \{V_\beta\}_{\beta \in B}$ be two open covers of a topological space X. We say $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$, and write $\mathfrak{U} \preceq \mathfrak{V}$, if there is a map $\phi : B \to A$ such that for each $\beta \in B$, we have $V_\beta \subseteq U_{\phi(\beta)}$. We call ϕ a *refinement map.*

Definition 1.2.7. We say a family $\{X_\alpha\}_{\alpha \in A}$ of subsets of a topological space X is *locally finite* if for every $x \in X$, there is an open neighbourhood V of x such that V intersects only finitely many of the X_α; that is, the set $\{\alpha \in A \mid V \cap X_\alpha \neq \emptyset\}$ is finite.

Definition 1.2.8. (Paracompact Spaces). A topological space X is called *paracompact* if every open cover $\mathfrak{U}$ of X has a locally finite refinement.

Theorem 1.2.9. *Manifolds are paracompact.*

Proof. See Theorem 1.15 of [6]. □

Definition 1.2.10. Let X be a topological space and $f : X \to \mathbb{R}$ a real-valued continuous function on X. The *support* of f is

$$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}},$$

where $\overline{S}$ denotes the closure of $S \subseteq X$ in X.

Definition 1.2.11. (Partitions of Unity). Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a manifold M . A *partition of unity subordinate to* $\mathfrak{U}$ is a collection $\{\rho_\alpha\}_{\alpha \in A}$ of smooth nonnegative functions such that

- for $\alpha \in A$, $\operatorname{supp} \rho_\alpha \subseteq U_\alpha$,
- the family $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in I}$ is locally finite,
- for $x \in M$, $\sum_{\alpha \in A} \rho_\alpha(x) = 1$.

Remark 1.2.12. What we have defined above is what is usually called a *smooth* partition of unity. Since we will only ever deal with manifolds, it will be convenient to drop the extra adjective.

Theorem 1.2.13. (Existence of Partitions of Unity). *Let M be a smooth manifold and $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ an open cover of M. Then there exists a partition of unity subordinate to $\mathfrak{U}$.*

Proof. See Theorem 2.23 of [6]. $\square$

2. Čech cohomology

In this section we will construct the Čech cohomology (with real coefficients) of a topological space X.

We start off by looking at open covers $\mathfrak{U}$ of X and how to build a simplicial complex $N(\mathfrak{U})$, called the *nerve* of the cover, using the data of finite intersections of open sets in $\mathfrak{U}$. This will give us a combinatorial model to approximate the topological space X. The question of when is this approximation "good enough" leads us to the notion of a *good cover* introduced by Weil in [3].

After defining the Čech cohomology of a cover $\mathfrak{U}$ by considering locally constant functions on each simplex $N(\mathfrak{U})$, we can then define the Čech cohomology of the space X; the main idea is that by repeatedly refining the cover $\mathfrak{U}$, $N(\mathfrak{U})$ will be an increasingly better approximation of X and so we define the Čech cohomology of X to be the Čech cohomology of a cover $\mathfrak{U}$ subject to repeated refinements in some sort of limiting process, which we make rigorous using the notion of a *direct limit*.

Keeping in mind the fact we will eventually want to compare the Čech and de Rham cohomologies of a *manifold* M, I have included some results specific to manifolds in the first subsections when dicussing nerves of open covers and good covers.

2.1. The Nerve of an Open Cover $\mathfrak{U}$.

Given an open cover $\mathfrak{U}$ of a topological space X, we now show how to construct a simplicial complex $N(\mathfrak{U})$, called the *nerve* of the cover; this is a combinatorial approximation of the underlying topological space X, which gets increasingly better as we refine the cover $\mathfrak{U}$.

Definition 2.1.1. (The Nerve of an Open Cover). Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a topological space X. For finite $J \subseteq A$, let $U_J := \bigcap_{j \in J} U_j$. Then the *nerve* of $\mathfrak{U}$ is

$$N(\mathfrak{U}) := \{J \subseteq A \mid J \text{ finite}, U_J \neq \emptyset\}.$$

By definition, $N(\mathfrak{U})$ consists only of nonempty finite subsets of A. Furthermore, note that $J' \subseteq J \subseteq A \Rightarrow U_J \subseteq U_{J'}$; thus $J \in N(\mathfrak{U}) \Rightarrow J' \in N(\mathfrak{U})$. Hence, the nerve of a cover $N(\mathfrak{U})$ is a simplicial complex; the vertices of $N(\mathfrak{U})$ are $\alpha \in A$ where $U_\alpha \neq \emptyset$.

Remark 2.1.2. If $J = \{\alpha_0, ..., \alpha_p\} \subseteq A$, we'll usually write $U_{\alpha_0 \cdots \alpha_p}$ instead of $U_{\{\alpha_0,...,\alpha_p\}}$. Note that in this notation, the order of the indices of U don't matter, e.g. $U_{\alpha\beta} = U_{\beta\alpha}$.

Remark 2.1.3. As defined, some may say the nerve $N(\mathfrak{U})$ of an open cover $\mathfrak{U}$ is the set of simplices of the abstract simpilcial complex $K = (V, N(\mathfrak{U}))$ where $V = \{i \in I \mid U_i \neq \emptyset\}$, rather than a simplicial complex itself. However, since two simplicial complexes are equal if and only if their set of simplices is equal, we will let $N(\mathfrak{U})$ denote both the set of simplices and the corresponding simplicial complex.

Example 2.1.4. (*The nerve of a cover of* S^1.) Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle and let $\mathfrak{U} = \{U_0, U_1, U_2\}$ be an open cover of S^1 by three open arcs of length $\frac{4\pi}{3}$:

$$U_0 := \left\{ e^{i\theta} \,\middle|\, \theta \in \left(0, \frac{4\pi}{3}\right) \right\},$$

$$U_1 := \left\{ e^{i\theta} \,\middle|\, \theta \in \left(\frac{2\pi}{3}, 2\pi\right) \right\},$$
$$U_2 := \left\{ e^{i\theta} \,\middle|\, \theta \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \right\}.$$

Then,

$$U_{01} = \left\{ e^{i\theta} \,\middle|\, \theta \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right) \right\},$$
$$U_{12} = \left\{ e^{i\theta} \,\middle|\, \theta \in \left(\frac{4\pi}{3}, 2\pi\right) \right\},$$
$$U_{02} = \left\{ e^{i\theta} \,\middle|\, \theta \in \left(0, \frac{2\pi}{3}\right) \right\},$$
$$U_{012} = \emptyset.$$

Hence, $N(\mathfrak{U}) = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$.

Remark 2.1.5. Notice that $N(\mathfrak{U})$ is a simplicial circle and so $N(\mathfrak{U})$ is a "good approximation" of S^1. Indeed, $\mathfrak{U}$ is what is known as a *good cover* of S^1.

2.2. **Good Covers.**

To define what a good cover is, we first need to know what it means for a topological space X to be *contractible*. Intuitively, this means that the space can be shrunk (through itself) to a single point. This idea is made rigorous by the notion of homotopy equivalence of topological spaces.

Definition 2.2.1. (Homotopic Maps). Let X, Y be two topological spaces and $f, g : X \to Y$ two continuous maps. f and g are said to be *homotopic* if there exists a continuous function $\gamma(x,t) : X \times [0,1] \to Y$ such that

1. $\forall x \in X, \gamma(x,0) = f(x)$,
2. $\forall x \in X, \gamma(x,1) = g(x)$.

Such a γ is called a *homotopy* between the two maps f and g.

Remark 2.2.2. Let M, N be manifolds, $f, g : M \to N$ two continuous maps, and $\gamma : M \times [0,1] \to N$ a homotopy between f and g. We can extend γ to a continuous map $\tilde{\gamma} : M \times \mathbb{R} \to N$ by defining

$$\tilde{\gamma}(x,t) := \begin{cases} \gamma(x,0) = f(x) & t < 0 \\ \gamma(x,t) & t \in [0,1] \\ \gamma(x,1) = g(x) & t > 1. \end{cases}$$

This is preferable for us because we now have a continuous map between two manifolds[2] rather than just two topological spaces.

Definition 2.2.3. (Homotopy Equivalence). Let X and Y be two topological spaces. X and Y are said to be *homotopy equivalent*, or *of the same homotopy type*, if there exists continuous maps $f : X \to Y, g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity on X and Y respectively.

If X has the homotopy type of a point, i.e. if the identity map id_X on X is homotopic to a constant map, then X is called *contractible*.

[2] $M \times \mathbb{R}$ is a product of manifolds and so is itself a manifold.

Example 2.2.4. Note that homotopy type is invariant under homeomorphism and so also under diffeomorphism. So if a manifold M is diffeomorphic to $\mathbb{R}^n$ for some $n \geq 0$, then M is contractible.

Example 2.2.5. Another source of simple examples of contractible spaces are star-like subsets of $\mathbb{R}^n$; that is, subsets $S \subseteq \mathbb{R}^n$ such that $\exists s_0 \in S$ such that $\forall s \in S$, we have

$$\{(1-t)s_0 + ts \mid t \in [0,1]\} \subseteq S.$$

Such a set S is called star-like with respect to s_0.

As it stands, there is the potential that when comparing the homotopy type of two manifolds we may need to consider maps that are continuous but not smooth. This is undesirable: when we think about maps between manifolds, we really only want to think about smooth maps between them. Fortunately, it turns out just considering smooth maps between manifolds is equivalent to considering all continuous maps between manifolds for the purposes of comparing homotopy type. This is thanks to the following result which we shall not prove:

Proposition 2.2.6. (Proposition 17.8 of [1]). *Every continuous map $f : M \to N$ between two manifolds is continuously homotopic to a smooth map.*

Definition 2.2.7. (Differentiable Retractions). (§1 of [3]) Let M be a contractible manifold. Then, by Remark 2.2.2 and Proposition 2.2.6, there is a smooth function $\gamma : M \times \mathbb{R} \to M$ such that

1. $\gamma(x,t) = x$ for $x \in M, t > 1$
2. the restriction $\gamma|_{M\times(-\infty,0]}$ of γ to $M \times (-\infty, 0]$ is constant

We call γ a *differentiable retraction* of M.

Definition 2.2.8. (Good Covers). (§1 of [3]) Let X be a topological space and $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of X. We say $\mathfrak{U}$ is a *good cover* of X if every nonempty finite intersection $U_J = \bigcap_{j\in J} U_J, J \subseteq A$ of open sets in $\mathfrak{U}$ is contractible; that is, for every $J \in N(\mathfrak{U})$, U_J is contractible.

Remark 2.2.9. In [1], the definition of a good cover $\mathfrak{U}$ of a manifold M of dimension n requires all nonempty intersections U_J to be diffeomorphic to $\mathbb{R}^n$. As noted in a previous remark, this implies that U_J is contractible. Hence, a good cover in [1] will also be a good cover for us.

Remark 2.2.10. Weil introduced the notion of a good cover in [3]. In §5 of the same paper, he also showed that when $X = M$ is a manifold and $\mathfrak{U}$ is a good cover of M, then M has the same homotopy type as (the geometric realisation of) $N(\mathfrak{U})$; we note that a related *nerve theorem* was shown a few years earlier by Karol Borsuk in [7].

Remark 2.2.11. If $K = (V, \Sigma)$ is an abstract simplicial complex (e.g. if K is the nerve of an open cover of some topological space), we will not usually distinguish K from its geometric realisation. When we do want to distinguish the two, we will use $|K|$ to denote the latter.

In Example 2.1.4, we had a good cover of S^1 whose nerve was was not only homotopy equivalent to S^1, but, also homeomorphic to S^1. It turns out that we can always find a good cover $\mathfrak{U}$ of a manifold M whose nerve $N(\mathfrak{U})$ is homeomorphic to

M. To show this, we first note that a triangulation of a manifold M gives us a good cover of M. All the heavy lifting of the result is then done using a theorem due to Stewart S. Cairns [8] which shows that all smooth manifolds are triangulable.

Proposition 2.2.12. (Triangulations give Good Covers). *Let X be a topological space. Let $K = (V, \Sigma)$ be a triangulation of X so that there is a homeomorphism $\varphi : |K| \xrightarrow{\sim} X$, where $|K|$ is the geometric realisation of K. For $v \in V$, let $U_v := \varphi(\mathrm{st}(v))$, where $\mathrm{st}(v)$ denotes the open star[3] of v in $|K|$. Then $\mathfrak{U} := \{U_v\}_{v \in V}$ is an open good cover of X which has nerve $N(\mathfrak{U}) = K$. In particular, all triangulable spaces admit good covers.*

Proof. Let J be a nonempty finite subset of V. Since φ is bijective, we have

$$U_J = \bigcap_{v \in J} \varphi(\mathrm{st}(v)) = \varphi\left(\bigcap_{v \in J} \mathrm{st}(v)\right).$$

If $J \notin \Sigma$, then there are $u, v \in J$ such that $\{u, v\} \notin \Sigma$. It follows then that

$$U_J \subseteq \mathrm{st}(v) \cap \mathrm{st}(u) = \emptyset,$$

and so $J \notin N(\mathfrak{U})$. On the other hand, if $J \in \Sigma$ (i.e. J is a simplex of K), the intersection $\bigcap_{v \in J} \mathrm{st}(v)$ is the union of the insides[4] of simplices which contain J as a sub-simplex, and so $U_J \supseteq \varphi(\mathrm{inside}(\Delta_J))$ is nonempty. Hence, $J \in N(\mathfrak{U})$. Thus, $N(\mathfrak{U}) = \Sigma$ and so, as simplicial complexes, $N(\mathfrak{U}) = K$.

To show that $\mathfrak{U}$ is a good cover, note that for $J \in N(\mathfrak{U}) = \Sigma$, we have

$$\bigcap_{v \in J} \mathrm{st}(v) = \bigcup_{\sigma \in \Sigma, J \subseteq \sigma} \mathrm{inside}(\Delta_\sigma) = \bigcup_{\sigma \in \Sigma, J \subseteq \sigma} (\mathrm{inside}(\Delta_\sigma) \cup \mathrm{inside}(\Delta_J))$$

Now for each simplex $\sigma \in \Sigma$ with $J \subseteq \sigma$, one sees that $\mathrm{inside}(\Delta_\sigma) \cup \mathrm{inside}(\Delta_J)$ is star-like with respect to points in $\mathrm{inside}(\Delta_J)$. Hence the above set is a union of sets which are star-like with respect to points in $\mathrm{inside}(\Delta_J)$, and so is contractible. Since homotopy type is invariant under homeomorphisms, we see that U_J is contractible. □

Theorem 2.2.13. (C^1-Manifolds are Triangulable). [8] *Let M be an n-dimensional C^1-manifold with an atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ such that the trivialisations $\varphi_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n$ are homeomorphisms onto $\mathbb{R}^n$. If A is finite, M admits a triangulation by a finite simplicial complex. Otherwise, M admits a triangulation by an infinite simplicial complex.*

Proof. The proof of this is beyond the scope of this text. I refer the interested reader to Cairns' original paper [8] on the matter. The reader may also find the paper [9] by J.H.C. Whitehead, which was written as a supplement to Cairns' paper, useful. □

Since smooth manifolds are obviously C^1-manifolds, the previous two results combine to give us:

[3] If v is a vertex of a simplicial complex $K = (V, \Sigma)$, then $\mathrm{st}(v) = \bigcup_{\sigma \in \Sigma, v \in \Sigma} \mathrm{inside}(\Delta_\sigma)$. For a simplex $\sigma \in K$, Δ_σ denotes the copy of σ in the geometric realisation of K. By the inside of a simplex, I mean the simplex without its faces. So, for example, the insde of a 2-simplex is the open triangle in the middle. With this convention, the inside of a vertex (i.e. a 0-simplex) is the the vertex itself.

[4] See footnote 3 at the bottom of the previous page.

Theorem 2.2.14. *Let M be a smooth manifold. Then there is a good open cover $\mathfrak{U}$ of M whose nerve $N(\mathfrak{U})$ has geometric realisation homeomorphic to M.*

Contained within this result, is the following important result:

Corollary 2.2.15. *A good cover of a manifold always exists.*

Alternative proofs of this corollary can be found in §1 of [3] and pp42-43 of [1]. Using triangulations, one can also show that for any open cover $\mathfrak{U}$ of a compact manifold M, there is a good open cover $\mathfrak{V}$ that is a refinement of $\mathfrak{U}$ (see Section 9). In part due to their more restrictive definition of a good cover (Remark 2.2.9), the argument of [1] shows something stronger than both of these results combined:

Corollary 2.2.16. (Corollary 5.2 of [1]) *Let M be an n-dimensional manifold and $\mathfrak{U}$ be an open cover of M. Then there is a good open cover $\mathfrak{V} = \{V_\beta\}_{\beta \in B}$ of M such that*

1. *$\mathfrak{V}$ is a refinement of $\mathfrak{U}$*
2. *Every nonempty finite intersection V_J is diffeomorphic to $\mathbb{R}^n$*

2.3. **Čech Cohomology of a Cover $\mathfrak{U}$.**

In this section and the next, we will construct Čech cohomology (with real coefficients) of a topological space X. As mentioned in the introduction, we will first construct the Čech cohomology of an open cover $\mathfrak{U}$ of X.

So, for this section, let us fix an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of X. For simplicity, we will further assume that the indexing set A is (totally) ordered with order $\leq$. We will understand $a < b$ to mean $a \leq b$ and $a \neq b$.

Definition 2.3.1. (Čech p-cochains). Let $p \geq 0$ be an integer. For $\alpha_0, ..., \alpha_p \in A$, let $\Gamma\left(U_{\alpha_0 \cdots \alpha_p}, \mathbb{R}\right)$ be the real vector space of locally constant real-valued functions $U_{\alpha_0 \cdots \alpha_p} \to \mathbb{R}$; if $U_{\alpha_0 \cdots \alpha_p} = \emptyset$, i.e. $\{\alpha_0, ..., \alpha_p\} \notin N(\mathfrak{U})$, we set $\Gamma\left(U_{\alpha_0 \cdots \alpha_p}, \mathbb{R}\right) = 0$. We then define the vector space of Čech p-cochains of $\mathfrak{U}$ to be the direct product

$$C^p(\mathfrak{U}, \mathbb{R}) := \prod_{\alpha_0 < \cdots < \alpha_p} \Gamma\left(U_{\alpha_0 \cdots \alpha_p}, \mathbb{R}\right).$$

For $p < 0$, it is natural to define $C^p(\mathfrak{U}, \mathbb{R}) := 0$.

Remark 2.3.2. An element $f \in C^p(\mathfrak{U}, \mathbb{R})$ has components $f_{\alpha_0 \cdots \alpha_p}$ where the indices are strictly increasing: $\alpha_0 < \cdots < \alpha_p$. We will more generally allow indices in any order (not just strictly increasing) as well as repetitions by adopting the convention that when two indices are interchanged, the sign changes:

$$f_{\dots\alpha\dots\beta\dots} = -f_{\dots\beta\dots\alpha}$$

Interpreting a permutation of the indices $\alpha_0, ..., \alpha_p$ as an orientation[5] of the p-simplex $\sigma = \{\alpha_0, ..., \alpha_p\} \in N(\mathfrak{U})_p$, we may think of $f \in C^p(\mathfrak{U}, \mathbb{R})$ as a tuple of locally constant real-valued functions indexed over the set of oriented p-simplices σ of $N(\mathfrak{U})$. In light of this, we could have alternatively defined the vector space of Čech p-cochains as

$$C^p(\mathfrak{U}, \mathbb{R}) := \prod_{\sigma \in N(\mathfrak{U})_p} \Gamma(U_\sigma, \mathbb{R}),$$

where $N(\mathfrak{U})_p$ denotes the set of p-simplices of $N(\mathfrak{U})$ and the total order on the set

[5] It is a fact that there are only two orientations of a p-simplex; we do not prove this here.

of vertices gives us a natural orientation of each simplex of $N(\mathfrak{U})$. In practice, the orientation of a simplex induced by the total order on the vertices isn't always the most natural orientation to use.

Definition 2.3.3. (The Čech Differential). Let

$$C^*(\mathfrak{U},\mathbb{R}) := \bigoplus_{p\geq 0} C^p(\mathfrak{U},\mathbb{R})$$

For $p \geq 0$, define $\delta : C^p(\mathfrak{U},\mathbb{R}) \to C^{p+1}(\mathfrak{U},\mathbb{R}); f \mapsto \delta f$ by setting

$$(\delta f)_{\alpha_0\cdots\alpha_{p+1}} := \sum_{k=0}^{p+1} (-1)^k f_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_{p+1}} \in \Gamma\Big(U_{\alpha_0\cdots\alpha_{p+1}},\mathbb{R}\Big)$$

where the caret denotes omission and the summands are to be replaced with their restrictions to $U_{\alpha_0\cdots\alpha_{p+1}}$. This defines a map $\delta : C^*(\mathfrak{U},\mathbb{R}) \to C^*(\mathfrak{U},\mathbb{R})$.

Remark 2.3.4. A calculation shows that the definition of δ respects our convention regarding indices; this is Exercise 8.4 of [1].

Lemma 2.3.5. $C^*(\mathfrak{U},\mathbb{R})$ *is a differential complex with differential operator* δ*. This differential complex is called* the Čech complex of the cover $\mathfrak{U}$.

Proof. For $p \geq 0, f \in C^p(\mathfrak{U},\mathbb{R})$ we have

$$\begin{aligned}
&\left(\delta^2 f\right)_{\alpha_0\cdots\alpha_{p+2}} = \sum_{k=0}^{p+2} (-1)^k (\delta f)_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_{p+2}} \\
&= \sum_{k=0}^{p+2} (-1)^k \left[\sum_{l=0}^{k-1} (-1)^l (f)_{\alpha_0\cdots\widehat{\alpha_l}\cdots\widehat{\alpha_k}\cdots\alpha_{p+2}} + \sum_{l=k+1}^{p+2} (-1)^{l-1} (f)_{\alpha_0\cdots\widehat{\alpha_k}\cdots\widehat{\alpha_l}\cdots\alpha_{p+2}}\right] \\
&= \sum_{l<k} (-1)^{k+l} (f)_{\alpha_0\cdots\widehat{\alpha_l}\cdots\widehat{\alpha_k}\cdots\alpha_{p+2}} + \sum_{l>k} (-1)^{k+l-1} (f)_{\alpha_0\cdots\widehat{\alpha_k}\cdots\widehat{\alpha_l}\cdots\alpha_{p+2}} \\
&= 0
\end{aligned}$$

Hence $\delta^2 = 0$. □

Definition 2.3.6. (Čech Cohomology of a Cover $\mathfrak{U}$). The *Čech cohomology of a cover* $\mathfrak{U}$ is the cohomology $\check{H}^*(\mathfrak{U},\mathbb{R})$ of the Čech complex of the cover; the p^{th} Čech cohomology group of $\mathfrak{U}$ is

$$\check{H}^p(\mathfrak{U},\mathbb{R}) = \frac{(\text{Ker } \delta) \cap C^p(\mathfrak{U},\mathbb{R})}{(\text{Im } \delta) \cap C^p(\mathfrak{U},\mathbb{R})}.$$

Remark 2.3.7. The Čech complex we have defined above is sometimes called the *alternating* Čech complex. One can also define a Čech complex without needing a total order on the indices by letting the direct product in Definition 2.3.1 be over *all* $p+1$-tuples of elements of A (in fact, this is what Weil does in [3]). It turns out these give rise to isomorphic cohomologies (see 01FG of [10]).

Example 2.3.8. (*Čech cohomology of a good cover of* S^1, Example 9.1 of [1]). Let $\mathfrak{U}$ be the open cover of S^1 given in Example 2.1.4. We notice that all finite intersections of the open sets of $\mathfrak{U}$ are connected. Hence, on each of the finite intersections, the vector space of locally constant real-valued functions is equal to the vector space of constant real-valued function and so isomorphic to $\mathbb{R}$ as a vector space. Hence, the Čech complex of $\mathfrak{U}$ is

$$\begin{array}{ccccccc} & & \mathbb{R}^3 & & \mathbb{R}^3 & & \\ & & \shortparallel & & \shortparallel & & \\ 0 & \longrightarrow & C^0(\mathfrak{U},\mathbb{R}) & \xrightarrow{\delta} & C^1(\mathfrak{U},\mathbb{R}) & \longrightarrow & 0 \\ & & (f_0,f_1,f_2) & \longmapsto & (f_1-f_0,f_2-f_1,f_0-f_2) & & \end{array}$$

We see that $\check{H}^0(\mathfrak{U},\mathbb{R}) = \operatorname{Ker}\delta \simeq \{(f_0,f_1,f_2) \in \mathbb{R}^3 \mid f_0 = f_1 = f_2\} \simeq \mathbb{R}$. By the rank-nullity theorem, $\operatorname{Im}\delta \simeq \mathbb{R}^2$ and so we deduce $\check{H}^1(\mathfrak{U},\mathbb{R}) \simeq \operatorname{Coker}\delta \simeq \mathbb{R}$.

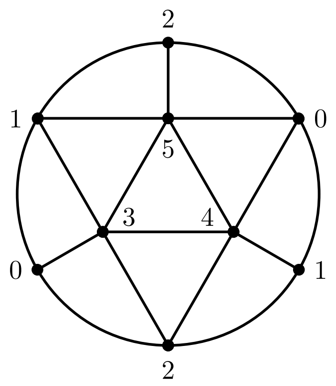
Note that all 1-cochains $\eta = (\eta_{01},\eta_{12},\eta_{20}) \in C^1(\mathfrak{U},\mathbb{R})$ are closed. If η is exact, then we see that $\eta_{01}+\eta_{12}+\eta_{20} = 0$. Hence, for example, we see that $\eta = (1,0,0)$ is a closed 1-cochain which is not exact.

Remark 2.3.9. (*Čech Complex from a Triangulation*). Let M be an n-dimensional smooth manifold and $K = (V,\Sigma)$ be a triangulation of M. Let $\mathfrak{U} = \{U_v\}_{v\in V}$ be the good open cover of M obtained by taking the open stars of the vertices of K as in Proposition 2.2.12. For each simplex $J \in N(\mathfrak{U})$, U_J is contractible (and so connected) so locally constant functions on U_J are constant; that is, $\Gamma(U_J,\mathbb{R}) \simeq \mathbb{R}$. Hence, letting $n_p := |K_p|$ be the number of p-simplices of K for $p \geq 0$, we see that

$$C^p(\mathfrak{U},\mathbb{R}) \simeq \mathbb{R}^{n_p}$$

For $p < 0$, one also sees that $C^p(\mathfrak{U},\mathbb{R}) = 0$.

Example 2.3.10. (*Čech cohomology of a good cover of* $\mathbb{RP}^2$, Exercise 9.10 of [1]). Consider the following triangulation of $\mathbb{RP}^2$, the real projective plane:



As dicussed in Proposition 2.2.12, by letting U_i be the open star of vertex i, this gives rise to a good cover $\mathfrak{U} = \{U_0,\ldots,U_5\}$ of $\mathbb{RP}^2$ whose nerve $N(\mathfrak{U})$ is precisely the triangulation shown above. By Remark 2.3.9, we then see that Čech complex $C^*(\mathfrak{U},\mathbb{R})$ of $\mathfrak{U}$ is

$$\begin{array}{ccccccccc} & & \mathbb{R}^6 & & \mathbb{R}^{15} & & \mathbb{R}^{10} & & \\ & & \shortparallel & & \shortparallel & & \shortparallel & & \\ 0 & \longrightarrow & C^0(\mathfrak{U},\mathbb{R}) & \xrightarrow{\delta_0} & C^1(\mathfrak{U},\mathbb{R}) & \xrightarrow{\delta_1} & C^2(\mathfrak{U},\mathbb{R}) & \longrightarrow & 0 \end{array}$$

As in Example 2.3.8, we see that $\check{H}^0(\mathfrak{U},\mathbb{R}) = \operatorname{Ker}\delta_0 \simeq \mathbb{R}$ and, by rank-nullity, $\operatorname{Im}\delta_0 = \mathbb{R}^5$. It remains to calculate $\check{H}^1(\mathfrak{U},\mathbb{R}) = \operatorname{Ker}\delta_1/\operatorname{Im}\delta_0$ and $\check{H}^2(\mathfrak{U},\mathbb{R}) = \operatorname{Coker}\delta_1$. To do this, we note that (ordering the bases of $C^1(\mathfrak{U},\mathbb{R})$ and $C^2(\mathfrak{U},\mathbb{R})$ "lexicographically") the map

$$\delta_1 : C^1(\mathfrak{U}, \mathbb{R}) \to C^2(\mathfrak{U}, \mathbb{R})$$
$$(\eta_{01}, \eta_{02}, \eta_{03}, \ldots, \eta_{45}) \mapsto ((\delta\eta)_{013}, (\delta\eta)_{014}, (\delta\eta)_{023}, \ldots, (\delta\eta)_{345})$$

is given by the matrix

$$\begin{pmatrix}
+1 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+1 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & +1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 & +1 & 0 & -1 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & -1 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & -1 & 0 & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & 0 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & +1
\end{pmatrix}$$

Using some row operations, one can then see that this matrix has rank 10. It follows that $\operatorname{Ker}\delta_1 \simeq \mathbb{R}^5$ and $\operatorname{Coker}\delta_1 \simeq 0$. Thus, we find the Čech cohomology of $\mathfrak{U}$ is:

$$\check{H}^n(\mathfrak{U}, \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To define the Čech cohomology of a topological space, we want to keep refining our open sets in some sort of limiting process to give ourselves the best chance of getting a good approximation of our space. Thus, it will be important for us to understand how the Čech cohomology of a cover of a topological space behaves under refinement. This is the topic of the following lemma.

Lemma 2.3.11. (Lemma 10.4.1 and Lemma 10.4.2 of [1] combined). *Let X be a topological space, and $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}, \mathfrak{V} = \{V_\beta\}_{\beta \in B}$ be two open covers of X with $\mathfrak{U} \preceq \mathfrak{V}$. Then refinement induces a unique map $\check{H}^*(\mathfrak{U}, \mathbb{R}) \to \check{H}^*(\mathfrak{V}, \mathbb{R})$ on Čech cohomology.*

Proof. Since $\mathfrak{U} \preceq \mathfrak{V}$, there is a refinement map $\phi : B \to A$ such that for all $\beta \in B$, we have $V_\beta \subseteq U_{\phi(\beta)}$. Given $\eta \in C^p(\mathfrak{U}, \mathbb{R})$, we can then define a cochain $\phi^\sharp \eta \in C^p(\mathfrak{V}, \mathbb{R})$ by setting

$$\left(\phi^\sharp \eta\right)_{\beta_0 \cdots \beta_p} := \eta_{\phi(\beta_0) \ldots \phi(\beta_p)}$$

This gives us a map $\phi^\sharp : C^*(\mathfrak{U}, \mathbb{R}) \to C^*(\mathfrak{V}, \mathbb{R})$ which is clearly also a homomorphism. In fact, $\phi^\sharp$ is a chain map[6] and thus induces a map on Čech cohomology $\overline{\phi^\sharp} : \check{H}^*(\mathfrak{U}, \mathbb{R}) \to \check{H}^*(\mathfrak{V}, \mathbb{R})$.

It remains to check that refinement gives us a *unique* map on cohomology. That is, given another refinement map $\psi : B \to A$, the induced map $\overline{\psi^\sharp}$ on cohomology is the same as $\overline{\phi^\sharp}$. To show this we will show the map given in Lemma 10.4.2 of [1], is a homotopy operator between $\psi^\sharp$ and $\phi^\sharp$; that is, we do Exercise 10.5 of [1]. The map in question is $K : C^p(\mathfrak{U}, \mathbb{R}) \to C^{p-1}(\mathfrak{V}, \mathbb{R})$ given by

$$(K\eta)_{\beta_0 \cdots \beta_p} = \sum_{i=0}^{p-1} (-1)^i \eta_{\phi(\beta_0) \ldots \phi(\beta_i) \psi(\beta_i) \ldots \psi(\beta_{p-1})}.$$

[6] This is an elementary calculation; see Lemma 10.4.1 of [1] for details.

i.e. we repeat each index β_i (once using ϕ and once using ψ), then take the alternating sum over the resulting terms. We have

$$\begin{aligned}(K(\delta\eta))_{\beta_0\ldots\beta_p} &= \sum_{j=0}^{p}(-1)^j(\delta\eta)_{\phi(\beta_0)\ldots\phi(\beta_j)\psi(\beta_j)\ldots\psi(\beta_p)}\\ &= \sum_{j>i}(-1)^{i+j}\eta_{\phi(\beta_0)\ldots\widehat{\phi(\beta_i)}\ldots\phi(\beta_j)\psi(\beta_j)\ldots\psi(\beta_p)}\\ &\quad+\sum_{j}(-1)^{2j}\eta_{\phi(\beta_0)\ldots\widehat{\phi(\beta_j)}\psi(\beta_j)\ldots\psi(\beta_p)}\\ &\quad+\sum_{j}(-1)^{2j+1}\eta_{\phi(\beta_0)\ldots\phi(\beta_j)\widehat{\psi(\beta_j)}\ldots\psi(\beta_p)}\\ &\quad+\sum_{j<i}(-1)^{j+i+1}\eta_{\phi(\beta_0)\ldots\phi(\beta_j)\psi(\beta_j)\ldots\widehat{\psi(\beta_i)}\ldots\psi(\beta_p)}\end{aligned}$$

where the sum over $j > i$ corresponds to omitting an index before the repeated β_j; the first sum over j corresponds to omitting the first copy of the repeated β_j; the second sum over j corresponds to omitting the second copy of the repeated β_j; and the sum over $j < i$ corresponds to omitting an index after the repeated β_j.

On the other hand, we have

$$\begin{aligned}(\delta(K\eta))_{\beta_0\ldots\beta_p} &= \sum_{i=0}^{p}(-1)^i(K\eta)_{\beta_0\ldots\widehat{\beta}_i\ldots\beta_p}\\ &= \sum_{j<i}(-1)^{i+j}\eta_{\phi(\beta_0)\ldots\phi(\beta_j)\psi(\beta_j)\ldots\widehat{\psi(\beta_i)}\ldots\psi(\beta_p)}\\ &\quad+\sum_{j>i}(-1)^{i+j-1}\eta_{\phi(\beta_0)\ldots\widehat{\phi(\beta_i)}\ldots\phi(\beta_j)\psi(\beta_j)\ldots\psi(\beta_p)}\end{aligned}$$

Adding these together, we get a telescoping sum:

$$\begin{aligned}((\delta K+K\delta)\eta)_{\beta_0\ldots\beta_p} &= \sum_{j=0}^{p}\left[\eta_{\phi(\beta_0)\ldots\widehat{\phi(\beta_j)}\psi(\beta_j)\ldots\psi(\beta_p)}-\eta_{\phi(\beta_0)\ldots\phi(\beta_j)\widehat{\psi(\beta_j)}\ldots\psi(\beta_p)}\right]\\ &= \sum_{j=0}^{p}\left[\eta_{\phi(\beta_0)\ldots\widehat{\phi(\beta_j)}\psi(\beta_j)\ldots\psi(\beta_p)}-\eta_{\phi(\beta_0)\ldots\phi(\beta_j)\widehat{\phi(\beta_{j+1})}\psi(\beta_{j+1})\ldots\psi(\beta_p)}\right]\\ &= \eta_{\psi(\beta_0)\ldots\psi(\beta_p)}-\eta_{\phi(\beta_0)\ldots\phi(\beta_p)}\\ &= \big((\psi^\sharp-\phi^\sharp)\eta\big)_{\beta_0\ldots\beta_p}\end{aligned}$$

Thus, $\delta K + K\delta = \psi^\sharp - \phi^\sharp$ and so K is a homotopy operator between $\psi^\sharp$ and $\phi^\sharp$.

□

2.4. **Direct Limits of Abelian Groups.**

The next (and final) step to constructing the Čech cohomology of a space is to formalise the limiting process that we want to do. This will be achieved by taking a *direct limit* over the directed set of covers of the topological space, ordered by refinement.

Definition 2.4.1. (Directed Sets). An (upward) *directed set* is a set A together with a reflexive and transitive relation (i.e. a preorder) $\leq$ on A which admits all (finite) upper bounds. A subset $B \subseteq A$ of a directed set A is called *cofinal* in A if for every $a \in A$, there is $b \in B$ such that $a \leq b$.

Example 2.4.2. The set of open covers of a topological space X is a directed set under refinement:

1. an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ is clearly a refinement of itself via the refinement map id_A
2. if $\mathfrak{U} \preceq \mathfrak{V} \preceq \mathfrak{W}$ with refinement maps ϕ, ψ, respectively, then $\mathfrak{U} \preceq \mathfrak{W}$ via the refinement map $\psi \circ \phi$ since "$\subseteq$" is transitive
3. Given any two open covers $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}, \mathfrak{V} = \{V_\beta\}_{\beta \in B}$, the cover $\mathfrak{W} = \{U_\alpha \cap V_\beta\}_{(\alpha,\beta) \in A \times B}$ is a refinement of $\mathfrak{U}$ and $\mathfrak{V}$ via the refinement maps $\mathrm{pr}_A : A \times B \to A; (a,b) \mapsto a$ and $\mathrm{pr}_B : A \times B \to B; (a,b) \mapsto b$.

By Corollary 2.2.16, the subset of good open covers of a manifold M is cofinal in the directed set of open covers of M.

Definition 2.4.3. (Direct Systems of Abelian Groups). A *direct system of abelian groups* is a collection $\{G_i\}_{i \in I}$ of groups indexed by a directed set I such that for any pair $a \leq b$ there is a group homomorphism $i_a^b : G_a \to G_b$ satisfying:

1. $i_a^a = \mathrm{id}_{G_a}$
2. $i_a^c = i_b^c \circ i_a^b$ whenever $a \leq b \leq c$.

Example 2.4.4. By Example 2.4.2, the set of open covers $\mathfrak{U}$ of a topological space X form a directed set under refinement. If $\mathfrak{U} \preceq \mathfrak{V}$, by Lemma 2.3.11 there is a (unique) homomorphism on Čech cohomology $i_{\mathfrak{U}}^{\mathfrak{V}} : \check{H}^*(\mathfrak{U}, \mathbb{R}) \to \check{H}^*(\mathfrak{V}, \mathbb{R})$ induced by refinement. Hence, the collection of (the underlying abelian groups of) Čech cohomology spaces $H^*(\mathfrak{U}, \mathbb{R})$ indexed by open covers $\mathfrak{U}$ of a topological space form a direct system of abelian groups.

Definition 2.4.5. (Direct Limit of a Direct System of Abelian Groups). ([1] p.112) Let $\{G_i\}_{i \in I}$ be a direct system of groups. Define an equivalence relation $\sim$ on the disjoint union $\coprod G_i$ by setting

$$g_a \sim g_b \iff i_a^c(g_a) = i_b^c(g_b) \text{ for some } a, b \leq c$$

where $g_a \in G_a, g_b \in G_b$. The *direct limit* of $\{G_i\}$, written $\varinjlim G_i$, is $\coprod G_i / \sim$, the quotient of the disjoint union by $\sim$; in other words, two elements of $\coprod G_i$ represent the same element in the direct limit if they are "eventually equal". We make the direct limit into a group by defining $[g_a] + [g_b] = [i_a^c(g_a) + i_b^c(g_b)]$, where $[g_a]$ is the equivalence class of g_a.

Intuitively, the role of direct limits in the construction of Čech cohomology is to allow us to "localise" information by only looking nearby a point. Another example of this phenomenon is when looking at *germs of smooth functions at a point.*

Definition 2.4.6. (Germs of Functions at a Point). Let M be a manifold and fix a point $x \in M$. Consider the set S of all pairs (f, U) where U is an open neighbourhood of x in M and $f : U \to \mathbb{R}$ is a smooth function. We define an equivalence relation $\sim$ on S by setting $(f, U) \sim (g, V)$ if and only if there is an open neighbourhood $W \subseteq U \cap V$ such that $p \in W$ and $f|_W = g|_W$. The set $C^\infty_{M,x}$ of germs of smooth functions at x is $S/\sim$, the set of equivalence classes of S under $\sim$. The equivalence class of (f, U) in $C^\infty_{M,x}$ is called the *germ* of f at x.

Example 2.4.7. Let us now see how to rephrase this definition using direct limits. Note that

- setting $U \leq V \Leftrightarrow V \subseteq U$, the set of open neighbourhoods $U \subseteq M$ of a point $x \in M$ clearly form a directed set;
- for each open neighbourhood U of x, the set $C^\infty(U)$ of smooth functions $f : U \to \mathbb{R}$ forms a vector space (and so an abelian group);
- whenever we have two open neighbourhoods $V \subseteq U$ of x, there is a restriction homomorphism $r_U^V : C^\infty(U) \to C^\infty(V)$; and
- the restriction homomorphisms satisfy 1. and 2. of Definition 2.4.5 (check!).

Thus, we can form the direct limit $\varinjlim C^\infty(U)$ over open neighbourhoods $U \subseteq M$ of x. Unravelling definitions, one sees that $\varinjlim C^\infty(U) = C^\infty_{M,x}$.

With the definition of direct limits, we are now finally ready to define the Čech cohomology of a topological space.

2.5. **Čech Cohomology of a Topological Space.**

Definition 2.5.1. (Čech Cohomology of a Topological Space) Let X be a topological space. The Čech cohomology of X is the direct limit

$$\check{H}^*(X, \mathbb{R}) := \varinjlim \check{H}^*(\mathfrak{U}, \mathbb{R})$$

where the direct limit is taken over the directed set of open covers $\mathfrak{U}$ of X ordered under refinement.

Remark 2.5.2. If X, Y are topological spaces and $f : X \xrightarrow{\sim} Y$ is a homeomorphism, then for every open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of X, we see that $f^{-1}\mathfrak{U} = \{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of Y.

As previously mentioned, we will only be looking at the case when X is the underlying topological space of a manifold M. We will briefly see in the next section how to define the *de Rham cohomology* $H^*_{\mathrm{dR}}(M)$ of an (orientable) manifold M using the differentiable structure on M. The next two results will be our main tools for calculating the Čech cohomology of a manifold M; we will prove them in a couple of sections once we have defined all the relevant objects.

Theorem 2.5.3. (The Čech-de Rham Isomorphism). *Let M be a manifold. The Čech cohomology of M (with coefficients in $\mathbb{R}$) is isomorphic to its de Rham cohomology:*

$$\check{H}^*(M, \mathbb{R}) \cong H^*_{\mathrm{dR}}(M)$$

Corollary 2.5.4. *Let M be a manifold and $\mathfrak{U}$ be a good cover of M. Then the Čech cohomology of M is isomorphic to the Čech cohomology of $\mathfrak{U}$, that is*

$$\check{H}(M, \mathbb{R}) \cong \check{H}(\mathfrak{U}, \mathbb{R})$$

Remark 2.5.5. The above Corollary is a special case of *Leray's Theorem.*

This corollary means that the calculations we did in the preceeding section in fact calculated the actual Čech cohomology of the manifold.

Example 2.5.6. By Example 2.3.8, we see that the Čech cohomology of the unit circle S^1 is

$$\check{H}^n(S^1,\mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } n = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

By Example 2.3.10, we see that the Čech cohomology of the real projective plane $\mathbb{RP}^2$ is

$$\check{H}^n(\mathbb{RP}^2,\mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.5.7. (*The Euler-Poincaré Characteristic*). Let M be an n-dimensional manifold which can be triangulated by a finite simplicial complex K. For $p \geq 0$, let n_p be the number of p-simplices of K. By assumption, each n_p is finite, and only finitely many n_p are nonzero. The *Euler-Poincaré Characteristic* of M can be defined as

$$\chi(M) := \sum_{p\in\mathbb{Z}} (-1)^p n_p$$

Recall from Proposition 2.2.12 that K gives rise to a good cover $\mathfrak{U}$ of M by taking open stars of the vertices of K. From Remark 2.3.9, we have for $p \in \mathbb{Z}$,

$$C^p(\mathfrak{U},\mathbb{R}) \cong \mathbb{R}^{n_p}.$$

By the rank-nullity theorem, we then see that we may rewrite $\chi(M)$ as

$$\begin{aligned}
\chi(M) &= \sum_{p\in\mathbb{Z}} (-1)^p \big[\dim_{\mathbb{R}}((\operatorname{Ker}\delta)\cap C^p(\mathfrak{U},\mathbb{R})) + \dim_{\mathbb{R}}\big((\operatorname{Im}\delta)\cap C^{p+1}(\mathfrak{U},\mathbb{R})\big)\big] \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim_{\mathbb{R}}((\operatorname{Ker}\delta)\cap C^p(\mathfrak{U},\mathbb{R})) \\
&\quad - \sum_{p\in\mathbb{Z}} (-1)^{p+1} \dim_{\mathbb{R}}\big((\operatorname{Im}\delta)\cap C^{p+1}(\mathfrak{U},\mathbb{R})\big) \\
&= \sum_{p\in\mathbb{Z}} (-1)^p [\dim_{\mathbb{R}}((\operatorname{Ker}\delta)\cap C^p(\mathfrak{U},\mathbb{R})) - \dim_{\mathbb{R}}((\operatorname{Im}\delta)\cap C^p(\mathfrak{U},\mathbb{R}))] \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim_{\mathbb{R}}\Big(\check{H}^p(\mathfrak{U},\mathbb{R})\Big).
\end{aligned}$$

Since $\mathfrak{U}$ is a good open cover of M, by Corollary 2.5.4, we have for $p \in \mathbb{Z}$,

$$\check{H}^p(M,\mathbb{R}) \cong \check{H}^p(\mathfrak{U},\mathbb{R}).$$

Hence, we may write the Euler-Poincaré characteristic of M in terms of the Čech cohomology of M:

$$\chi(M) = \sum_{p\in\mathbb{Z}} (-1)^p \dim_{\mathbb{R}}\Big(\check{H}^p(M,\mathbb{R})\Big).$$

By Theorem 2.5.3, we may also write $\chi(M)$ in terms of the de Rham cohomology of M.

3. Brief Summary of de Rham Cohomology

In this section we will do a very brief summary of differential forms, which can be thought of as a generalisation of vector calculus on $\mathbb{R}^3$, and the de Rham cohomology theory. The main reference when writing this section was [1], though one can find a more complete account of the material presented in [5] or [6].

3.1. The de Rham Complex on $\mathbb{R}^n$.

Definition 3.1.1. (The de Rham Complex on $\mathbb{R}^n$). ([1] p.13) Let $x_1, ..., x_n$ be the standard coordinate functions on $\mathbb{R}^n$. We define Ω^* to be the algebra over $\mathbb{R}$ generated by $dx_1, ...dx_n$ with the relations

$$\begin{cases} (dx_i)^2 = 0 \\ dx_i dx_j = -dx_j dx_i. \end{cases}$$

As a vector space over $\mathbb{R}$, Ω^* has basis

$$1, \underset{i<j}{dx_i, dx_i dx_j}, \underset{i<j<k}{dx_i dx_j dx_k}, ..., dx_1 \cdots dx_n.$$

The C^∞ differential forms on $\mathbb{R}^n$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*.$$

Thus, if ω is such a form, then ω can be uniquely written as $\sum f_{i_1 \dots i_q} dx_{i_1} \cdots dx_{i_1}$ where the coefficients $f_{i_1 \dots i_q}$ are C^∞ functions. Using multi-index notation, we may also write $\omega = \sum f_{(i)} dx_{(i)}$. The algebra $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$ is naturally graded, here $\Omega^q(\mathbb{R}^n)$ is the vector space of C^∞ q-forms on $\mathbb{R}^n$. There is a differential operator $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$, called the *exterior derivative* defined as follows:

1. if $f \in C^0(\mathbb{R}^n)$, then $df = \sum \frac{\partial f}{\partial x_i} dx_i$
2. if $\omega = \sum f_{(i)} dx_{(i)}$, then $d\omega = \sum df_{(i)} dx_{(i)}$.

Proposition 3.1.2. $d^2 = 0$*; that is, the de Rham complex on* $\mathbb{R}^n$ *is a differential complex.*

Proof. By linearity of d, to show $d^2 = 0$, it is suffices to show that $d^2\omega = 0$ for all monomials $\omega = f dx_{i_1} \cdots dx_{i_q} \in \Omega^q(\mathbb{R}^n)$. For such an ω, this essentially follows from the symmetry of second partial derivatives for C^∞ functions:

$$\begin{aligned} d^2\omega &= d\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j dx_{i_1} \cdots dx_{i_q}\right) \\ &= \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j x_k} dx_k dx_j dx_{i_1} \cdots dx_{i_q} \\ &= \sum_{k \neq j} \frac{\partial^2 f}{\partial x_j \partial x_k} dx_k dx_j dx_{i_1} \cdots dx_{i_q} \qquad \left(\text{since } (dx_j)^2 = 0\right) \\ &= \sum_{k<j} \left(\frac{\partial^2 f}{\partial x_j \partial x_k} - \frac{\partial^2 f}{\partial x_k \partial x_j}\right) dx_k dx_j dx_{i_1} \cdots dx_{i_q} \\ &= 0 \end{aligned}$$

$\square$

Definition 3.1.3. (de Rham Cohomology of $\mathbb{R}^n$). The *de Rham cohomology* $H^*_{\mathrm{DR}}(\mathbb{R}^n)$ of $\mathbb{R}^n$ is the cohomology of $\Omega^*(\mathbb{R}^n)$; the q^{th} de Rham cohomology of $\mathbb{R}^n$ is

$$H^q_{\mathrm{DR}}(\mathbb{R}^n) = \frac{\operatorname{Ker} d \cap \Omega^q(\mathbb{R}^n)}{\operatorname{Im} d \cap \Omega^q(\mathbb{R}^n)}.$$

de Rham q-cocycles are called *closed q-forms* and de Rham q-coboundaries are called *exact q-forms.*

Definition 3.1.4. (Wedge Product of Forms). Let $\omega = \sum f_{(i)} dx_{(i)}, \tau = \sum g_{(j)} dx_{(j)}$ be two differential forms on $\mathbb{R}^n$. Then the *wedge product* of ω and τ is

$$\omega \wedge \tau := \sum f_{(i)} g_{(j)} dx_{(i)} dx_{(j)}$$

From a basic calculation, one sees that the wedge product makes $\Omega^*(\mathbb{R}^n)$ into a graded real algebra which is commutative in the graded sense:

$$\omega \wedge \tau = (-1)^{\deg \omega \deg \tau} \tau \wedge \omega$$

Proposition 3.1.5. (Proposition 1.3 of [1]). *With respect to the wedge product, d is an antiderivation. That is, we have*

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge (d\omega)$$

Remark 3.1.6. (p.15 of [1]) Note that everything done in this section works equally over any open subset $U \subseteq \mathbb{R}^n$. For instance, we may define

$$\Omega^*(U) := \{C^\infty \text{ functions on } U\} \otimes_{\mathbb{R}} \Omega^*.$$

Thus, we can also analagously define the de Rham cohomology $H^*_{\mathrm{DR}}(U)$ of U.

3.2. The de Rham Complex on a Manifold.

We'll now generalise the construction above to arbitrary manifolds following §2 of [1]. Only the bare essentials will be included here and we will freely use some basic language from category theory. The idea is that for all (n-dimensional) manifolds, the local situation is of $\mathbb{R}^n$ and so all definitions carry over. Firstly, note that a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ induces a pullback map on forms

$$f^* : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^m); \omega \mapsto \omega \circ f.$$

Letting $\Omega^*(f) := f^*$, one can show the following:

Proposition 3.2.1. (p.20 of [1]). *The de Rham Complex Ω^* is a contravariant functor from the category of Euclidean spaces and smooth maps to the category of commutative differential graded algebras and their homomorphisms.*

Remark 3.2.2. Contained in the above proposition is the statement that f^* commutes with the exterior derivative d. So, for example, if $x_1, ..., x_n$ are local coordinates in the neighbourhood of some point, then any form can be written as $\sum_{(i)} g_{(i)} dx_{i_1} \cdots dx_{i_q}$ in this neighbourhood, and we have

$$f^*\left(\sum g_{(i)} dx_{i_1} \cdots dx_{i_q}\right) = \sum \left(g_{(i)} \circ f\right) df_{i_1} \cdots df_{i_q}$$

where $f_{i_j} = x_{i_j} \circ f = f^*\left(x_{i_j}\right)$ is the $i_j{}^{\text{th}}$ component of f. Using the chain rule, one can then also show that the formula for the exterior derivative d is independent of the coordinate system.

Definition 3.2.3. (The de Rham Complex on a Manifold). (p.21 of [1]) Let M be a manifold with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$. A *differential form* ω on M is a collection of forms ω_{U_α} on each U_α which are compatible in the following sense: if i and j are inclusions

$$\begin{tikzcd} & U \\ U \cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & \\ & V \end{tikzcd}$$

then $i^*\omega_U = j^*\omega_V$ in $\Omega^*(U \cap V)$.

By functoriality of Ω^*, the exterior derivative and the wedge product extend to differential forms on a manifold, giving us the de Rham complex $\Omega^*(M)$ of M. Just as for $\mathbb{R}^n$, for a smooth map $f : M \to N$, $\Omega^*(f) = f^* : \Omega^*(N) \to \Omega^*(M)$ is the natural pullback. In this way, we extend Ω^* to a contravariant functor on the category of smooth manifolds.

Remark 3.2.4. If M is a manifold, $\Omega^q(M)$ is the vector space of differential q-forms on M, as was the case when $M = \mathbb{R}^n$. Recall that if $U \subseteq M$ is an open subset of M, then U is also a manifold; $\Omega^q(U)$ is the vector space of differential q-forms on U; if $U = \emptyset$, we set $\Omega^q(U) := 0$.

Definition 3.2.5. (de Rham Cohomology of a Manifold). Let M be a manifold. The *de Rham cohomology* $H^*_{\mathrm{DR}}(M)$ of M is the cohomology of $\Omega^*(M)$; the q^{th} de Rham cohomology of M is

$$H^q_{\mathrm{DR}}(M) := \frac{(\operatorname{Ker} d) \cap \Omega^q(M)}{(\operatorname{Im}\, d) \cap \Omega^q(M)}.$$

As was the case for $\mathbb{R}^n$, de Rham q-cocycles are called *closed q-forms* and de Rham q-coboundaries are called *exact q-forms*.

3.3. **Integration of Forms.**

We can integrate top forms with compact support. Let U be an open subset of $\mathbb{R}^n$ and $\omega = f(x)dx_1 \cdots dx_n$ be a compactly supported n-form on U. We define

$$\int_U \omega = \int_U f dx_1 \cdots dx_n := \int_U f \,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

where the right-hand-side is the Riemann integral of f over U, following the notation of [1]. The order of the x_i's matter when integrating a form: if $\pi \in S_n$ is a permutation, then

$$\int_U f dx_{\pi(1)} \cdots dx_{\pi(n)} = (\operatorname{sgn} \pi) \int_U f dx_1 \cdots dx_n$$

If U, V are open subsets of $\mathbb{R}^n$ and $T : U \to V$ is a change of coordinates; i.e., a diffeomorphism $U \xrightarrow{\sim} V$,

$$(x_1, ..., x_n) = T(y_1, ..., y_n) = (T_1(y_1, ..., y_n), ..., T_n(y_1, ..., y_n)),$$

then one can show that

$$dT_1 \cdots dT_n = J(T) dy_1 ... dy_n$$

where $J(T)$ is the Jacobian determinant of T (see Exercise 3.1 of [1] and/or Exercise 3.7 of [5]). Abusing notation slightly to write $x_i = T_i$, we may write the above as

$$dx_1 \cdots dx_n = \frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)} dy_1 \cdots dy_n$$

By the usual change of variables formula from multivariable calculus, one then sees that

$$\int_V T^*\omega = \int_U (f \circ T) dT_1 \cdots dT_n = \pm \int_U \omega$$

where the sign is the sign of $J(T)$. This leads us to the following definition:

Definition 3.3.1. (Orientation-Preserving Diffeomorphisms). A diffeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ is called *orientation-preserving* if $J(T) > 0$; that is, if its Jacobian determinant is positive at every point in $\mathbb{R}^n$. Orientation-preserving diffeomorphisms are the diffeomorphisms under which the integrals of forms are invariant.

Definition 3.3.2. (Oriented Manifolds). If is a manifold M, we say an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M is *oriented* if all the transition functions $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ are orientation-preserving. We say M is *oriented* if it admits an oriented atlas.

For surfaces (i.e. a 2-manifold) in $\mathbb{R}^3$, you might be familiar with the fact that a surface is oriented if it admits a "continuous choice of outward facing normal", which is obtained by the cross-product of two continuous choices of independent tangent vectors. This fact has a generalistion to n-dimensions which we won't prove:

Proposition 3.3.3. (Proposition 3.2 of [1]). *A manifold M of dimension n is orientable if and only if it admits a global nowhere vanishing n-form.*

Definition 3.3.4. (Orientation of a Connected Manifold). (p.29 of [1]) If M is a connected n-dimensional manifold, then any two global nonwhere vanishing n -forms ω and ω' differ by a nonvanishing global smooth function $\omega = f\omega'$, where f is strictly positive or strictly negative. We define an equivalence relation on n-forms on M by saying ω and ω' are *equivalent* if and only if f is strictly positive. This splits $\Omega^n(M)$ into two equivalence classes; the equivalence classes are called *orientations* on M. We write $[M]$ for an orientation on M.

Let M be an orientable connected n-dimensional manifold with oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$ and fix an orientation $[M]$ on M. Pick a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. We define the integral of a compactly supported top form $\omega \in \Omega^n(M)$ by

$$\int_{[M]} \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega$$

where $\int_{U_\alpha} \rho_\alpha \omega = \int_{\varphi^{-1}(U_\alpha)} (\rho_\alpha \omega) \circ \varphi_\alpha^{-1}$. Note that $\operatorname{supp} \rho_\alpha \omega$ is a closed subset of the compact set $\operatorname{supp} \omega$ and so is itself compact. One can show that the definition of $\int_{[M]} \omega$ is indepedent of the choice of oriented atlas $\{(U_\alpha, \rho_\alpha)\}$ and partition of unity $\{\rho_\alpha\}$ (Proposition 3.3 of [1]). When an orientation of M is understood, we will sometimes write $\int_M \omega$ instead of $\int_{[M]} \omega$.

Definition 3.3.5. (Manifolds with boundary). (p.30 of [1]) A manifold of dimension n with boundary is a topological space M together with a set of ordered pairs $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that

1. $M \subseteq \bigcup_{\alpha \in A} U_\alpha$
2. For each $\alpha \in A$, φ_α is a homeomorphism of U_α with either $\mathbb{R}^n$ or the closed upper half plane $\mathbb{H}^n = \{(x_1, ..., x_n \mid x_n \geq 0)\}$.
3. The transition functions $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}$ are C^∞-maps.

The *boundary* ∂M of M is an $(n-1)$-dimensional manifold; it is the union of the preimages $\varphi_\alpha^{-1}(\{x_n = 0\})$ where $\varphi_\alpha : U_\alpha \cong \mathbb{H}^n$.

As in Definition 1.2.1, the pairs $(U_\alpha, \varphi_\alpha)$ are called (coordinate) *charts*, and φ_α is called the *trivialisation* of U_α; the cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ is sometimes called a *trivialising cover* of M; and the datum $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is called an *atlas* of M.

An oriented atlas for a manifold M with boundary induces, in a natural way, an oriented atlas on its boundary ∂M. The orientation conventions we use are to make writing Stokes' Theorem down easier, which we now state:

Theorem 3.3.6. (Stokes' Theorem). (Theorem 3.5 of [1]) *If ω is an $(n-1)$-form with compact support on an oriented manifold M of dimension n and if ∂M is given the induced orientation, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

4. The Čech-de Rham isomorphism

This section is an exposition of the proof of the Čech-de Rham Isomorphism (Theorem 2.5.3) given by Weil in §2 of [3] though I have chosen to make some changes to the notation.[7] Let us first give a brief outline of Weil's proof.

Let us fix an oriented n-dimensional manifold M with oriented atlas $\{(V_\beta, \varphi_\beta)\}_{\beta \in B}$. Let us also fix a good open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M. We will then define the Čech-de Rham double complex associated to the cover $\mathfrak{U}$ together with the horizontal differential δ and vertical differential d of this complex. We then augment the double complex to include the Čech complex of $\mathfrak{U}$ (as the -1^{th} row of the double complex) and the de Rham complex of M (as the -1^{th} column of the double complex).

The differentials (together with the augmentation maps) makes each of the rows and columns of the (augmented) double complex into differential complexes; we will show that these complexes are exact. We then use the exactness of the rows and columns to show that the Čech cohomology $\check{H}^*(\mathfrak{U}, \mathbb{R})$ of the cover $\mathfrak{U}$ is isomorphic to the de Rham cohomology $H^*_{\mathrm{DR}}(M)$ of M.

Since $\check{H}^*(\mathfrak{U}, \mathbb{R})$ is defined as a direct limit over open covers ordered by refinement, and since the set of good covers of M is cofinal (Corollary 2.2.16), this will lead to a proof of Theorem 2.5.3.

4.1. **The Augmented Čech-de Rham Double Complex.**

Definition 4.1.1. (The Čech-de Rham Double Complex). Let $p, q \geq 0$ be integers. A *differential coelement of bidegree* (p, q) is an element $\xi = \left(\xi_{\alpha_0 \cdots \alpha_p}\right)$ of the vector space

$$C^{p,q}(\mathfrak{U}, \mathbb{R}) := \prod_{\alpha_0 < \ldots < \alpha_p} \Omega^q\left(U_{\alpha_0 \cdots \alpha_p}\right).$$

So ξ is a tuple of differential q-forms on each (nonempty) $p+1$-fold intersection of open sets of $\mathfrak{U}$.

The vertical differential of ξ is defined by $d\xi := (d\xi_\sigma)$, which is an element of bidegree $(p, q+1)$; similarly, the horizontal differential of ξ is defined by $\delta\xi := (\delta\xi_\sigma)$, which is an element of bidegree $(p+1, q)$.

Remark 4.1.2. As in Remark 2.3.2, we will allow indices in any order by taking ξ to be an alternating function of its indices; we may thus interpret $\xi = (\xi_\sigma)$ as a tuple of differential q-forms indexed over the set of the (oriented) p-simplices σ of the nerve $N(\mathfrak{U})$ and could have alternatively defined

$$C^{p,q}(\mathfrak{U}, \mathbb{R}) := \prod_{\sigma \in N(\mathfrak{U})_p} \Omega^q(U_\sigma)$$

Remark 4.1.3. Essentially the same calculations as in Lemma 2.3.5 and Proposition 3.1.2 show that $\delta^2 = 0$ and $d^2 = 0$, respectively. Thus, each row and each column of the Čech-de Rham double complex is a differential complex. We also note that that d and δ commute: $d\delta = \delta d$.

[7] A differential element of bidegree (p, q) for us will be an *alternating* coelement of bidegree (q, p) in [3] (so the complex in [3] is the transpose of ours). Also, Weil uses m instead of q for the degree of a form. For us: ω will denote a globally defined differential form on M; ξ and ζ will both denote differential coelements; η will denote a Čech cochain; and γ will denote a differentiable retraction, reserving φ for trivialisations from an atlas. Furthermore, we use r for the inclusion $\Omega^m(M) \hookrightarrow C^{0,1}(\mathfrak{U}, \mathbb{R})$ instead of d.

We will now augment the Čech-de Rham double complex to include the Čech complex $C^*(\mathfrak{U},\mathbb{R})$ of $\mathfrak{U}$ and the de Rham complex $\Omega^*(M)$ of M.

Firstly, note that for each $p \geq 0$, $\xi = \left(\xi_{\alpha_0\cdots\alpha_p}\right) \in C^{p,0}(\mathfrak{U},\mathbb{R})$ has $d\xi = 0$ if and only if each $\xi_{\alpha_0\cdots\alpha_p}$ is a locally constant real-valued function on $U_{\alpha_0\cdots\alpha_p}$. That is, $d\xi = 0$ if and only if ξ is equal to a Čech p-cochain $\eta \in C^p(\mathfrak{U},\mathbb{R})$. Thus,

$$C^p(\mathfrak{U},\mathbb{R}) = \operatorname{Ker}(d : C^{p,0}(\mathfrak{U},\mathbb{R}) \to C^{p,1}(\mathfrak{U},\mathbb{R})),$$

and we may think of $C^p(\mathfrak{U},\mathbb{R})$ as the -1^{th} row of the Čech-de Rham double complex using the inclusion $C^p(\mathfrak{U},\mathbb{R}) \hookrightarrow C^{p,0}(\mathfrak{U},\mathbb{R})$ as the augmentation map.

Now for $q \geq 0$, let $\omega \in \Omega^q(M)$ be a differential q-form on M. By taking the restrictions of ω to each of the U_α, we get a differential coelement $\xi = (\xi_\alpha) := \left(\omega|_{U_\alpha}\right)$ of bidegree $(0,q)$. This gives us a map $r : \Omega^q(M) \to C^{0,q}(\mathfrak{U},\mathbb{R}); \omega \mapsto r\omega$ where $r\omega$ is the differential coelement ξ defined above. Note that r is clearly injective, and we have $\delta(r\omega) = 0$. On the other hand, if a differential coelement $\xi = (\xi_\alpha) \in C^{0,q}(\mathfrak{U})$ of bidegree $(0,q)$ has $\delta\xi = 0$, then ξ agrees on each of the double overlaps, and so $\xi = (\xi_\alpha) = \left(\omega|_{U_\alpha}\right)$ for some globally defined q-form ω on M to each U_α. Thus,

$$\Omega^q(M) \simeq \operatorname{Ker}(\delta : C^{0,q}(\mathfrak{U},\mathbb{R}) \to C^{1,q}(\mathfrak{U},\mathbb{R})),$$

and we may think of $\Omega^*(M)$ as the -1^{th} column of the Čech-de Rham double complex using the injective map r as the augmentation map.

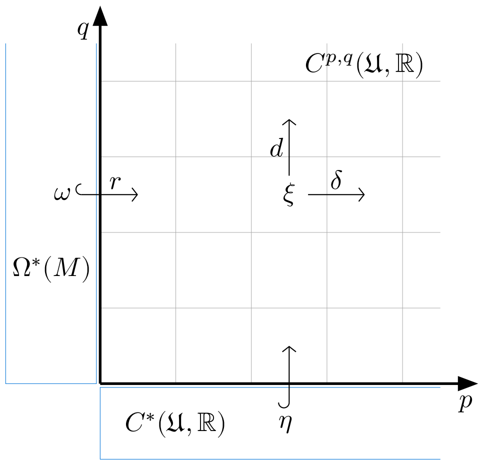


FIGURE. The Augmented Čech-de Rham complex with the Čech complex $C^*(\mathfrak{U},\mathbb{R})$ as the -1^{th} row and the de Rham complex $\Omega^*(M)$ as the -1^{th} column.

Remark 4.1.4. Note that, even in the Augmented Čech-de Rham complex, two steps in the positive p direction and two steps in the positive q direction are both still 0; the rows and columns of the augmented double complex are also differential complexes.

4.2. **Exactness of the Rows.**

We will now show that the nonnegative part of each row of the double complex is exact by exhibiting a homotopy operator between the identity map and the zero map on the nonnegative part of each row. Along the lines of Example 1.1.9, one sees that this will show that the cohomology of the rows is zero in nonnegative dimension. Note that the results of this section hold for *any* cover $\mathfrak{U}$ of M, not just good covers.

Let us first fix a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$.

Definition 4.2.1. (Hamilton's Homotopy Operator).[8] Let q be a nonnegative integer. Define $K : C^{0,q}(\mathfrak{U}, \mathbb{R}) \to \Omega^q(M)$ by

$$K\xi = K(\xi_\alpha) := \sum_{\alpha\in A} \rho_\alpha \xi_\alpha \in \Omega^q(M)$$

For $p > 0$, define $K : C^{p,q}(\mathfrak{U}, \mathbb{R}) \to C^{p-1,q}(\mathfrak{U}, \mathbb{R})$ by

$$(K\xi)_{\alpha_0\cdots\alpha_{p-1}} := \sum_{\alpha\in A} \rho_\alpha \xi_{\alpha\alpha_0\cdots\alpha_{p-1}} \in \Omega^q\big(U_{\alpha_0\cdots\alpha_{p-1}}\big)$$

To see why K is well-defined, note that for each $\alpha \in A$, if ξ_α is a form on U_α, multiplying by ρ_α gives us a form $\rho_\alpha\xi_\alpha$ with support contained in U_α. We can then extend $\rho_\alpha\xi_\alpha$ to a globally defined form on M by defining it to be zero on $M \setminus U_\alpha$. Similarly, if $\xi_{\alpha\alpha_0\cdots\alpha_{p-1}}$ is a form defined on $U_{\alpha\alpha_0\cdots\alpha_{p-1}} \subseteq U_\alpha$, multiplying by ρ_α gives us a form $\rho_\alpha\xi_{\alpha\alpha_0\cdots\alpha_{p-1}}$ with support in $U_{\alpha\alpha_0\cdots\alpha_{p-1}}$ which we can then extend to a form defined on the larger set $U_{\alpha_0\cdots\alpha_{p-1}}$ by defining it to be zero on $U_{\alpha_0\cdots\alpha_{p-1}} \setminus U_\alpha$.

Lemma 4.2.2. *For $p, q \geq 0$, $\xi \in C^{p,q}(\mathfrak{U}, \mathbb{R})$, we have $\xi = K\delta\xi + \delta K\xi$.*

$$\xi = \begin{cases} K\delta\xi + \delta K\xi & p > 0 \\ K\delta\xi + rK\xi & p = 0 \end{cases}$$

Proof. For $p > 0$, we have

$$\begin{aligned}
(K\delta\xi)_{\alpha_0\cdots\alpha_p} &= \sum_{\alpha\in A} \rho_\alpha (\delta\xi)_{\alpha\alpha_0\cdots\alpha_p} \\
&= \sum_{\alpha\in A} \rho_\alpha \left[\xi_{\alpha_0\cdots\alpha_p} + \sum_{k=0}^{p} (-1)^{k+1} \xi_{\alpha\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p}\right] \\
&= \xi_{\alpha_0\cdots\alpha_p} + \sum_{\alpha\in A}\sum_{k=0}^{p} (-1)^{k+1} \rho_\alpha \xi_{\alpha\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p} \\
&= \xi_{\alpha_0\cdots\alpha_p} - \sum_{\alpha\in A}\sum_{k=0}^{p} (-1)^{k} \rho_\alpha \xi_{\alpha\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p} \\
(\delta K\xi)_{\alpha_0\cdots\alpha_p} &= \sum_{k=0}^{p} (-1)^k (K\xi)_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p} \\
&= \sum_{k=0}^{p} (-1)^k \sum_{\alpha\in A} \rho_\alpha \xi_{\alpha\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p} \\
&= \sum_{k=0}^{p}\sum_{\alpha\in A} (-1)^k \rho_\alpha \xi_{\alpha\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p} \\
&= \sum_{\alpha\in A}\sum_{k=0}^{p} (-1)^k \rho_\alpha \xi_{\alpha\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p}
\end{aligned}$$

where the summands are to be replaced with their restrictions to $U_{\alpha_0\cdots\alpha_p}$(as in Definition 2.3.3). In the last equality we were allowed to swap the sums because at each point, only finitely many of the ρ_α are nonzero (see Definition 1.2.11). Adding the two together, we see that $\xi = K\delta\xi + \delta K\xi$ as desired.

[8]In a footnote of [3], Weil attributes the homotopy operator K to Norman Tyson Hamilton, one of Weil's PhD students whilst he was at the University of Chicago.

For $p = 0$, we similarly have

$$(K\delta\xi)_{\alpha_0} = \sum_{\alpha\in A} \rho_\alpha (\delta\xi)_{\alpha\alpha_0} = \sum_{\alpha\in A} \rho_\alpha\big(\xi_{\alpha_0} - \xi_\alpha\big) = \xi_{\alpha_0} - \sum_{\alpha\in A} \rho_\alpha \xi_\alpha$$
$$(rK\xi)_{\alpha_0} = (K\xi)_{\alpha_0} = \sum_{\alpha\in A} \rho_\alpha \xi_\alpha$$

where, as above, the summands are to be replaced with their restrictions to U_{α_0}. We see that $\xi = K\delta\xi + rK\xi$ as desired. □

Thus, if $\xi \in C^{p,q}(\mathfrak{U}, \mathbb{R})$ is a differential coelement such that $\delta\xi = 0$, then

$$\xi = \begin{cases} \delta K\xi & p > 0 \\ rK\xi & p = 0. \end{cases}$$

Hence, we see that the cohomology of the q^{th} row is 0 in nonnegative dimension; since $\Omega^q(M) \simeq \operatorname{Ker}\big(\delta : C^{0,q}(\mathfrak{U}, \mathbb{R}) \to C^{1,q}(\mathfrak{U}, \mathbb{R})\big)$ the cohomology of the q^{th} row is in dimension -1. We additionally note if $\omega \in \Omega^q(M)$ is a differential q-form on M, then $\omega = Kr\omega$.

4.3. **Exactness of the Columns (The Poincaré Lemma).**

In a similar fashion, we will show that the nonnegative part of each column of the double complex is exact complex by exhibiting a homotopy operator between the identity map and the zero map on the nonnegative part of each column. Unlike the previous section, the results of this section do depend on the cover $\mathfrak{U}$ being a good cover.

We will first construct the splitting map on the nonnegative part of the de Rham complex $\Omega^*(U)$ of an arbitrary contractible open subset U of M. This, together with the fact that U is contractible (and so connected), will show that the de Rham cohomology of U is $\mathbb{R}$ in dimension 0 and is 0 in dimension greater than 0, a result known as the *Poincaré Lemma.*[9] Since the components of a differentiable coelement are differential forms on contractible open subsets of M, we can then define a splitting map on differential coelements by defining it to be that splitting map componentwise.

So suppose that $U \subseteq M$ is an open subset of M which is contractible, and so admits a differentiable retraction $\gamma : U \times \mathbb{R} \to U$. Recall from Definition 2.2.7 that γ is constant on $U \times (-\infty, 0]$; let $a \in U$ be this constant value. Let $q > 0$ and suppose $\omega \in \Omega^q(U)$ is a differential q-form on U. For a point in $U \cap U_\beta$ $(\beta \in B)$, we have local coordinates $x_1, ..., x_n$ on $U \cap U_\alpha$ furnished by the trivialisation $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ (so $\varphi_\beta = (x_1, ..., x_n)$). Then, for $x \in U \cap U_\beta$ the pullback $\gamma^*\omega \coloneqq \omega \circ \gamma$ of ω by γ may then be written in terms of the local coordinates as

$$\gamma^*\omega(x, t) = \sum_{(i)} f_{(i)}(x, t) dx_{i_1} \cdots dx_{i_q} + \sum_{(j)} g_{(j)}(x, t) dt dx_{j_1} \cdots dx_{j_{q-1}}, \qquad (1)$$

where (i) and (j) are multi-indices of length q and $q - 1$ respectively. We define a form $I\omega$ of degree $q - 1$ by

$$I\omega \coloneqq \sum_{(j)} \left(\int_0^1 g_{(j)}(x, t)\, \mathrm{d}t \right) dx_{j_1} \cdots dx_{j_{q-1}}$$

Since the atlas is oriented, the calculations in §3.3 show that I is invariant under change of coordinates and hence the formula aboves defines a differential form $I\omega$

[9] This is usually proven using *homotopy invariance of de Rham cohomology.*

of degree $q-1$ on all of U. This gives us a map $I : \Omega^q(U) \to \Omega^{q-1}(U)$ for $q > 0$. For $q = 0$, we define $I\omega = 0$.

Lemma 4.3.1. *Let $\omega \in \Omega^q(U)$. Then*

$$\omega = \begin{cases} Id\omega + dI\omega & \text{if } q > 0 \\ Id\omega + \omega(a) & \text{if } q = 0 \end{cases}$$

where a is the constant value of γ.

Proof. Let $q > 0$. Since d commutes with pullbacks (Remark 3.2.2), we locally have

$$\begin{aligned}
\gamma^*(d\omega) &= d(\gamma^*\omega) \\
&= d\left[\sum_{(i)} f_{(i)}(x,t)dx_{i_1}\cdots dx_{i_q} + \sum_{(j)} g_{(j)}(x,t)dtdx_{j_1}\cdots dx_{j_{q-1}}\right] \\
&= \sum_{(i)}\left(\frac{\partial f_{(i)}}{\partial t}dt + \sum_{k=1}^{n}\frac{\partial f_{(i)}}{\partial x_k}dx_k\right)dx_{i_1}\cdots dx_{i_q} \\
&\quad + \sum_{(j)}\left(\frac{\partial g_{(j)}}{\partial t}dt + \sum_{k=1}^{n}\frac{\partial g_{(j)}}{\partial x_k}dx_k\right)dtdx_{j_1}\cdots dx_{j_{q-1}}
\end{aligned}$$

Since $(dt)^2 = 0$ and $dx_k dt = -dtdx_k$, we then see that

$$I(d\omega) = \sum_{(i)}\left(\int_0^1 \frac{\partial f_{(i)}}{\partial t}\,\mathrm{dt}\right)dx_{i_1}\cdots dx_{i_q} - \sum_{(j)}\left(\sum_{k=1}^{n}\left(\int_0^1 \frac{\partial g_{(j)}}{\partial x_k}\,\mathrm{dt}\right)dx_k\right)dx_{j_1}\cdots dx_{j_{q-1}}$$

On the other hand, by the Leibniz integral rule, we have

$$\begin{aligned}
d(I\omega) &= d\left[\sum_{(j)}\left(\int_0^1 g_{(j)}(x,t)\,\mathrm{dt}\right)dx_{j_1}\cdots dx_{j_{q-1}}\right] \\
&= \sum_{(j)}\left(\sum_{k=1}^{n}\left(\int_0^1 \frac{\partial g_{(j)}}{\partial x_k}\,\mathrm{dt}\right)dx_k\right)dx_{j_1}\cdots dx_{j_{q-1}}
\end{aligned}$$

Hence, the Fundamental Theorem of Calculus gives

$$\begin{aligned}
Id\omega + dI\omega &= \sum_{(i)}\left(\int_0^1 \frac{\partial f_{(i)}}{\partial t}\,\mathrm{dt}\right)dx_{i_1}\cdots dx_{i_q} \\
&= \sum_{(i)}\left[f_{(i)}(x,1) - f_{(i)}(x,0)\right]dx_{i_1}\cdots dx_{i_q}
\end{aligned}$$

By an explicit calculation (moved to Section 8) one can check that the above is indeed equal to ω.

For $q = 0$, $\omega = f(x)$ is a smooth function on U, and so

$$\gamma^* d\omega = d(\gamma^*\omega) = d(f\circ\gamma) = \frac{\partial(f\circ\gamma)}{\partial t}dt + \sum_{k=1}^{n}\frac{\partial(f\circ\gamma)}{\partial x_k}dx_k$$

Then by the Fundamental Theorem of Calculus

$$Id\omega(x) = \int_0^1 \frac{\partial(f\circ\gamma)}{\partial t}(x,t)\,\mathrm{dt} = (f\circ\gamma)(x,1) - (f\circ\gamma)(x,0) = \omega(x) - \omega(a)$$

giving the claim. $\square$

So, if $n > 0$ and $\omega \in \Omega^n(U)$ has $d\omega = 0$, then $\omega = Id\omega + dI\omega = dI\omega$ and so we see that $H^n_{\mathrm{DR}}(U) = 0$ for $n > 0$. Closed 0-forms are real-valued locally continuous functions on U and so, since U is connected, $H^0_{\mathrm{DR}}(U) \simeq \mathbb{R}$. We have thus shown:

Proposition 4.3.2. (Poincaré Lemma). *Let $U \subseteq M$ be a contractible open subset of an oriented manifold. Then the de Rham cohomology of U is*

$$H^n_{\mathrm{DR}}(U) = \begin{cases} \mathbb{R} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We now define the map I on differential coelements by defining it to be the above map componentwise. Using the data of a good cover, we fix for each simplex $\sigma \in N(\mathfrak{U})$, a differentiable retraction $\gamma_\sigma : U_\sigma \times \mathbb{R} \to U_\sigma$. Recall from Definition 2.2.7 that γ_σ is constant on $U_\sigma \times (-\infty, 0]$; we will denote this constant value by a_σ. We can thus define $I_\sigma : \Omega^*(U_\sigma) \to \Omega^{*-1}(U_\sigma)$ for each simplex $\sigma \in N(\mathfrak{U})$ as above.

Definition 4.3.3. (Integration Homotopy Operator). Let p be a nonnegative integer. Define $I : C^{p,0}(\mathfrak{U}, \mathbb{R}) \to C^p(\mathfrak{U}, \mathbb{R})$ by

$$I\xi := 0$$

For $q > 0$, define $I : C^{p,q}(\mathfrak{U}, \mathbb{R}) \to C^{p,q-1}(\mathfrak{U}, \mathbb{R})$ by $(I\xi)_\sigma := (I_\sigma \xi_\sigma)$

Since everything is defined componentwise, the identities of Lemma 4.3.1 carry over and so we see that for $\xi \in C^{p,q}(\mathfrak{U}, \mathbb{R})$, $p \geq 0$

$$\xi = \begin{cases} Id\xi + dI\xi & \text{if } q > 0 \\ Id\xi + \xi(a) & \text{if } q = 0 \end{cases}$$

where $\xi(a) = (\xi(a_\sigma)) \in C^{p,0}(\mathfrak{U}, \mathbb{R})$ if $\xi \in C^{p,q}(\mathfrak{U}, \mathbb{R})$. As in the previous section, we thus see that the cohomology of the p^{th} column is zero.

4.4. The Čech-de Rham Isomorphism.

We will now show that the Čech cohomology $\check{H}^*(\mathfrak{U}, \mathbb{R})$ of $\mathfrak{U}$ and the de Rham cohomology $H^*_{\mathrm{DR}}(M)$ of M are isomorphic. The way we do this is by "travelling along the anti-diagonals of the Čech-de Rham double complex" using the maps δ, d, K and I. The approach we will use is the one presented by Weil in [3]. An alternative (but morally equivalent) approach is to compare the *total cohomology* of the Čech-de Rham double complex to $\check{H}^*(\mathfrak{U}, \mathbb{R})$ and $H^*_{\mathrm{DR}}(M)$ (see §8 − 9 of [1]).

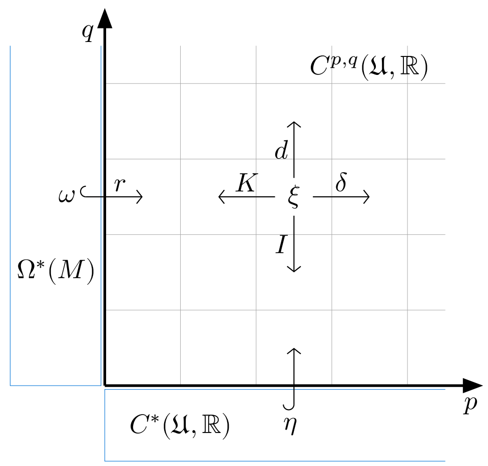


Figure. The Augmented Čech-de Rham complex with the maps δ, d, K and I.

Weil's approach is to consider tuples of coelements satisfying certain equations, we shall call these *Weil staircases* (this is not standard terminology).

Definition 4.4.1. (Weil Staircases). A *Weil staircase of degree* m is a tuple $\left(\omega, \xi^{(0)}, \xi^{(1)}, ..., \xi^{(m-1)}, \eta\right)$ where

- $\omega \in \Omega^m(M)$ is a differential form of degree $m > 0$ on M
- $\xi^{(h)} \in C^{h,m-h-1}(\mathfrak{U}, \mathbb{R})$ is a differential coelement of bidegree $(h, m-h-1)$ for $0 \leq h \leq m-1$
- $\eta \in C^m(\mathfrak{U}, \mathbb{R})$ a Čech m-cochain of $\mathfrak{U}$

such that the following equations are satisfied

1. $r\omega = d\xi^{(0)}$
2. $\delta\xi^{(h)} = d\xi^{(h+1)}$ for $0 \leq h \leq m-2$
3. $\delta\xi^{(m-1)} = \eta$

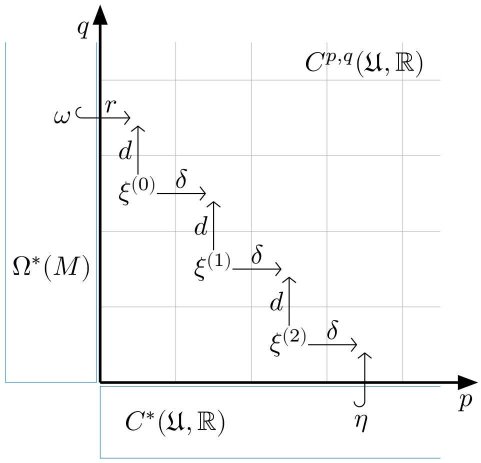


Figure. A Weil staircase of degree 3 on the double complex.

Remark 4.4.2. Note that if $\left(\omega, \xi^{(0)}, \xi^{(1)}, ..., \xi^{(m-1)}, \eta\right)$ is a Weil staircase of degree $m > 0$, then

$$\begin{aligned} d\delta\xi^{(h)} &= 0 \qquad (0 \leq h \leq m-1) \\ \delta\eta &= 0 \\ \delta d\omega &= 0 \end{aligned}$$

Also, since $d\omega$ is a form on M, we further see that $d\omega = K\delta d\omega = 0$. Setting

$$\begin{aligned} \mathfrak{F}_m &:= (\operatorname{Ker} d) \cap \Omega^m(M) \\ \mathfrak{H}_m &:= (\operatorname{Im} d) \cap \Omega^m(M) \\ \mathfrak{F}_{m,h} &:= (\operatorname{Ker} d\delta) \cap C^{h,m-h-1}(\mathfrak{U}, \mathbb{R}) \qquad (0 \leq h \leq m-1) \\ \mathfrak{H}_{m,h} &:= \left(\operatorname{Ker}\, d \cap \mathfrak{F}_{m,h}\right) + \left(\operatorname{Ker} \delta \cap \mathfrak{F}_{m,h}\right) \leq \mathfrak{F}_{m,h} \end{aligned}$$

we see the that we have $\omega \in \mathfrak{F}_m$, and $\xi^{(h)} \in \mathfrak{F}_{m,h}$ for $0 \leq h \leq m-1$.

So, we know that the elements of a Weil staircase are contained in the vector spaces $\mathfrak{F}_{m,h}$. The following lemma shows that, conversely, if a coelement $\xi^{(h)}$ lies in $\mathfrak{F}_{m,h}$, then $\xi^{(h)}$ lies on a Weil staircase of degree m. Furthermore, the lemma shows how travelling along a Weil staircase gives us isomorphisms between the quotient spaces $\mathfrak{F}_{m,h}/\mathfrak{H}_{m,h}$.

Lemma 4.4.3. (§2 of [3]) *Let $m > 0$ and $0 \leq h < m-1$. Then the equation $\delta\xi^{(h)} = d\xi^{(h+1)}$ gives an isomorphism $\frac{\mathfrak{F}_{m,h}}{\mathfrak{H}_{m,h}} \simeq \frac{\mathfrak{F}_{m,h+1}}{\mathfrak{H}_{m,h+1}}$ which may be explicitly written as*

$$\frac{\mathfrak{F}_{m,h}}{\mathfrak{H}_{m,h}} \xrightarrow{\sim} \frac{\mathfrak{F}_{m,h+1}}{\mathfrak{H}_{m,h+1}}; \xi^{(h)} + \mathfrak{H}_{m,h} \mapsto \xi^{(h+1)} := I\delta\xi^{(h)} + \mathfrak{H}_{m,h+1}$$

with inverse

$$\frac{\mathfrak{F}_{m,h+1}}{\mathfrak{H}_{m,h+1}} \xrightarrow{\sim} \frac{\mathfrak{F}_{m,h}}{\mathfrak{H}_{m,h}}; \xi^{(h+1)} + \mathfrak{H}_{m,h+1} \mapsto \xi^{(h)} := Kd\xi^{(h+1)} + \mathfrak{H}_{m,h}$$

Proof. Let us first check that the map $\frac{\mathfrak{F}_{m,h}}{\mathfrak{H}_{m,h}} \to \frac{\mathfrak{F}_{m,h+1}}{\mathfrak{H}_{m,h+1}}$ given above is well-defined. If $\xi^{(h)} \in \mathfrak{F}_{m,h} \leq C^{h,m-h-1}(\mathfrak{U}, \mathbb{R})$, then $d\delta\xi^{(h)} = 0$, and by Lemma 4.3.1 we see that $\xi^{(h+1)} := I\delta\xi^{(h)} \in C^{h+1,m-h-2}(\mathfrak{U}, \mathbb{R})$ has

$$d\xi^{(h+1)} = d\big(I\delta\xi^{(h)}\big) = \delta\xi^{(h)} - Id\big(\delta\xi^{(h)}\big) = \delta\xi^{(h)}$$

so that $\xi^{(h+1)} = I\delta\xi^{(h)}$ solves the equation $\delta\xi^{(h)} = d\xi^{(h+1)}$ and lies in $\mathfrak{F}_{m,h+1}$. If $\xi^{(h)} \in \mathfrak{H}_{m,h}$, then by definition of $\mathfrak{H}_{m,h}$

$$\xi^{(h)} = X + Y$$

for some $X, Y \in \mathfrak{F}_{m,h} \leq C^{h,m-h-1}(\mathfrak{U}, \mathbb{R})$ where $dX = 0, \delta Y = 0$. Now since we have $m - h - 1 > 0$ and $dX = 0$, Lemma 4.3.1 gives $X = dIX$ and thus

$$\delta\xi^{(h)} = \delta X + 0 = \delta dIX = d(\delta IX)$$

Hence, $d\big(I\delta\xi^{(h)} - \delta IX\big) = \delta\xi^{(h)} - \delta\xi^{(h)} = 0$ so that

$$\xi^{(h+1)} = I\delta\xi^{(h)} = \big(I\delta\xi^{(h)} - \delta IX\big) + \delta IX \in \mathfrak{H}_{m,h+1}$$

Thus, the map $\frac{\mathfrak{F}_{m,h}}{\mathfrak{H}_{m,h}} \to \frac{\mathfrak{F}_{m,h+1}}{\mathfrak{H}_{m,h+1}}$ is well-defined and solves the equation. It is clear that any other solution $\zeta^{(h+1)}$ must lie in the coset $\xi^{(h+1)} + \mathfrak{H}_{m,h+1}$.

Similarly , given $\xi^{(h+1)} \in \mathfrak{F}_{m,h+1}$ one sees by Lemma 4.2.2 that $\xi^{(h)} := Kd\xi^{(h+1)}$ solves the equation $\delta\xi^{(h)} = d\xi^{(h+1)}$ and that $\xi^{(h)} \in \mathfrak{F}_{m,h}$. If $\xi^{(h+1)} \in \mathfrak{H}_{m,h+1}$, then as before

$$\xi^{(h+1)} = X + Y$$

for some $X, Y \in \mathfrak{F}_{m,h+1} \leq C^{h+1,m-h-2}(\mathfrak{U}, \mathbb{R})$ where $dX = 0, \delta Y = 0$. Since we have $m - h - 2 \geq 0$ and $\delta Y = 0$, Lemma 4.2.2 gives $Y = \delta KY$ and thus

$$d\xi^{(h+1)} = 0 + dY = d\delta KY = \delta(dKY)$$

Hence, $\delta\big(Kd\xi^{(h+1)} - dKY\big) = d\xi^{(h+1)} - d\xi^{(h+1)} = 0$ so that

$$\xi^{(h)} = Kd\xi^{(h+1)} = dKY + \big(Kd\xi^{(h+1)} - dKY\big) \in \mathfrak{H}_{m,h+1}$$

Thus, the map $\frac{\mathfrak{F}_{m,h+1}}{\mathfrak{H}_{m,h+1}} \to \frac{\mathfrak{F}_{m,h}}{\mathfrak{H}_{m,h}}$ is also well-defined and solves the equation. As before, it is clear that any other solution $\zeta^{(h)}$ must lie in the coset $\xi^{(h)} + \mathfrak{H}_{m,h}$.

Since both maps solve the same equation, we see that they are inverses of each other. In particular, they are both isomorphisms. □

In a similar fashion, the following two lemmas show how equations 1. and 3. of Definition 4.4.1 gives us isomorphisms $H^m_{\mathrm{DR}}(M) = \mathfrak{F}_m/\mathfrak{H}_m \simeq \mathfrak{F}_{m,0}/\mathfrak{H}_{m,0}$ and $\breve{H}^m(\mathfrak{U}, \mathbb{R}) \simeq \mathfrak{F}_{m,m-1}/\mathfrak{H}_{m,m-1}$ respectively; we may think of them as instructions on how to "get on/off" a Weil staircase; their proofs are similar to the previous lemma, so they are omitted and left as exercises for the reader.

Lemma 4.4.4. (§2 of [3]) *Let $m > 0$. Then the equation $\delta\omega = d\xi^{(0)}$ gives an isomorphism*

$$\frac{\mathfrak{F}_{m,0}}{\mathfrak{H}_{m,0}} \xrightarrow{\sim} \frac{\mathfrak{F}_m}{\mathfrak{H}_m}; \xi^{(0)} + \mathfrak{H}_{m,0} \mapsto \omega := Kd\xi^{(0)} + \mathfrak{H}_m$$

with inverse

$$\frac{\mathfrak{F}_m}{\mathfrak{H}_m} \xrightarrow{\sim} \frac{\mathfrak{F}_{m,0}}{\mathfrak{H}_{m,0}}; \omega \mapsto \xi^{(0)} + \mathfrak{H}_m := Ir\omega + \mathfrak{H}_{m,0}$$

Lemma 4.4.5. (§2 of [3]) *Let $m > 0$. Then the equation $\delta\xi^{(m-1)} = \eta$ gives an isomorphism*

$$\frac{\mathfrak{F}_{m,m-1}}{\mathfrak{H}_{m,m-1}} \xrightarrow{\sim} \check{H}^m(\mathfrak{U}, \mathbb{R}); \xi^{(m-1)} + \mathfrak{H}_{m,m-1} \mapsto [I\delta\xi]$$

with inverse

$$\check{H}^m(\mathfrak{U}, \mathbb{R}) \xrightarrow{\sim} \frac{\mathfrak{F}_{m,m-1}}{\mathfrak{H}_{m,m-1}}; [\eta] \mapsto K\eta + \mathfrak{H}_{m,m-1}$$

Chaining all of these isomorphisms together, we get

Theorem 4.4.6. (The Čech-de Rham Isomorphism for Good Covers). (§2 of [3]) *Let M be an oriented manifold and $\mathfrak{U}$ be a good cover of M. Then*

$$H^*_{\mathrm{DR}}(M) \simeq \check{H}^*(\mathfrak{U}, \mathbb{R}).$$

If ω is a closed m-form on M, $m \geq 0$, then the Čech cohomology class corresponding to $[\omega] \in H^m_{\mathrm{DR}}(M)$ under the above isomorphism is respresented by

$$\eta = (\delta I)^m (r\omega)$$

If η is a Čech m-cocycle, $m \geq 0$, then the de Rham cohomology class corresponding to $[\eta] \in \check{H}^m(\mathfrak{U}, \mathbb{R})$ under the above isomorphism is represented by

$$\omega = K(dK)^m \eta$$

Remark 4.4.7. For $m = 0$, we can use a "degenerate" Weil staircase and solve the equation $r\omega = \eta$ to get an isomorphism.

Remark 4.4.8. From Theorem 2.5.3, we may deduce that the Čech cohomologies of any two good covers $\mathfrak{U}, \mathfrak{V}$ of an oriented manifold M are isomorphic.

Given a Čech m-cocycle $\eta = \left(\eta_{\alpha_0 \cdots \alpha_m}\right)$, where $m > 0$, we can use induction to expand the second formula in Theorem 4.4.6 to get the following expression stated in §2 of [3]:

$$\omega = (-1)^{\frac{m(m-1)}{2}} \sum_{\alpha_0, \ldots, \alpha_m} \eta_{\alpha_0 \cdots \alpha_m} \rho_{\alpha_m} d\rho_{\alpha_0} \wedge \cdots \wedge d\rho_{\alpha_{m-1}}$$

Since any cover may be refined to a good cover (Corollary 2.2.16) and the Čech cohomology of M is defined as a direct limit over open covers ordered under refinement, to show Theorem 2.5.3, one just needs to check that either of the formulas in Theorem 4.4.6 are compatible with restrictions; this follows from the fact that refinement commutes with both d and δ. This completes the proof of Theorem 2.5.3. From this we also immediately deduce Corollary 2.5.4.

5. Some Calculations on Surfaces

5.1. The Circle S^1.

Using induction and the Mayer-Vietoris Sequence (see Exercise 4.3 of [1]), one may calculate the de Rham cohomology of the n-sphere for $n \geq 1$ and find

$$H^k_{\mathrm{DR}}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

We will first look at S^1. We consider the good open cover $\mathfrak{U} = \{U_0, U_1, U_2\}$ given in Example 2.1.4, together with the natural trivialisations

$$\varphi_0 : U_0 \xrightarrow{\sim} \left(0, \frac{4\pi}{3}\right)$$
$$\varphi_1 : U_1 \xrightarrow{\sim} \left(\frac{2\pi}{3}, 2\pi\right)$$
$$\varphi_2 : U_2 \xrightarrow{\sim} \left(\frac{-2\pi}{3}, \frac{2\pi}{3}\right)$$

which one can check makes $\{(U_\alpha, \varphi_\alpha)\}$ into an oriented C^∞-atlas for S^1. One sees that $(\operatorname{Ker} d) \cap \Omega^0(S^1) = (\operatorname{Ker} \delta) \cap C^0(\mathfrak{U}, S^1)$ so nothing interesting happens to realise $H^0_{\mathrm{DR}}(S^1) \simeq \check{H}^0(\mathfrak{U}, \mathbb{R})$. Let us now look at the Čech-de Rham isomorphism in dimension one: $H^1_{\mathrm{DR}}(S^1) \simeq \check{H}^1(\mathfrak{U}, \mathbb{R})$. To use the formulas of Theorem 4.4.6, we either need to pick a partition of unity subordinate to $\mathfrak{U}$ (to go from Čech to de Rham), or pick a differentiable retraction for each simplex σ of the nerve $N(\mathfrak{U})$ (to go from de Rham to Čech).

Let us first do Čech to de Rham, and pick a partition of unity $\{\rho_0, \rho_1, \rho_2\}$ subordinate to $\mathfrak{U}$. Then one sees that, if $\eta = (\eta_{01}, \eta_{12}, \eta_{20}) = (1, 0, 0)$ is the non-trivial Čech cocycle we found in Example 2.3.8, then the form obtained from Theorem 4.4.6 which represents the class in $H^1_{\mathrm{DR}}(S^1)$ corresponding to $[\eta]$ is a form supported in $U_0 \cap U_1$ with integral -1 (see Example 9.9 of [1]).

Let us now do de Rham to Čech. The simplest scenario in this case is if for each simplex $\sigma \in N(\mathfrak{U})$ we consider the differentiable retraction γ_σ whose restriction to $U_\sigma \times [0, 1]$ is one of the following:

$$U_0 \times [0,1] \to U_0; \left(e^{i\theta_0}, t\right) \mapsto \exp(it\theta_0 + i(1-t)a_0) \qquad \theta_0, a_0 \in \left(0, \frac{4\pi}{3}\right)$$
$$U_1 \times [0,1] \to U_1; \left(e^{i\theta_1}, t\right) \mapsto \exp(it\theta_1 + i(1-t)a_1) \qquad \theta_1, a_1 \in \left(\frac{2\pi}{3}, 2\pi\right)$$
$$U_2 \times [0,1] \to U_2; \left(e^{i\theta_2}, t\right) \mapsto \exp(it\theta_2 + i(1-t)a_2) \qquad \theta_2, a_2 \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$$
$$U_{01} \times [0,1] \to U_{01}; \left(e^{i\theta_{01}}, t\right) \mapsto \exp(it\theta_{01} + i(1-t)a_{01}) \qquad \theta_{01}, a_{01} \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$$
$$U_{12} \times [0,1] \to U_{12}; \left(e^{i\theta_{12}}, t\right) \mapsto \exp(it\theta_{12} + i(1-t)a_{12}) \qquad \theta_{12}, a_{12} \in \left(\frac{4\pi}{3}, 2\pi\right)$$
$$U_{20} \times [0,1] \to U_{20}; \left(e^{i\theta_{20}}, t\right) \mapsto \exp(it\theta_{20} + (1-t)a_{20}) \qquad \theta_{20}, a_{20} \in \left(0, \frac{2\pi}{3}\right)$$

where the a_σ are fixed for $\sigma \in N(\mathfrak{U})$. This determines the integration operator I defined in the previous section. For S^1, there is a canonical representative of the generator of $H^1_{\mathrm{DR}}(S^1)$, namely $\frac{d\theta}{2\pi}$; from Theorem 4.4.6, a Čech cocycle which represents the Čech cohomology class corresponding to $\left[\frac{d\theta}{2\pi}\right]$ is $\eta := (\delta I r)\left(\frac{d\theta}{2\pi}\right)$ which we may calculate to be:

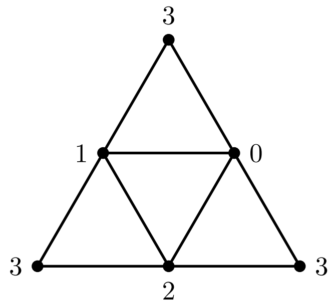
$$\eta_{01} = \frac{\theta_1 - a_1}{2\pi} - \frac{\theta_0 - a_0}{2\pi} = \frac{a_0 - a_1}{2\pi}$$
$$\eta_{12} = \frac{\theta_2 - a_2}{2\pi} - \frac{\theta_1 - a_1}{2\pi} = -1 + \frac{a_1 - a_2}{2\pi}$$
$$\eta_{20} = \frac{\theta_0 - a_0}{2\pi} - \frac{\theta_2 - a_2}{2\pi} = \frac{a_2 - a_0}{2\pi}$$

In fact, by the Fundamental Theorem of Calculus, we see that even for more general differentiable retractions, the η you will get out of the formula will be of the above form (with potentially also a different constant to -1 depending on the choice of atlas). One also sees that the above discussion will generalise to any good open cover whose nerve is a simplicial circle.

Remark 5.1.1. We note that the cocycles we obtained were off by a sign from what we intuitively might expect. For example, Čech cocyle is the negative of what we might expect it to be: if we choose our retractions so that the constant points are $a_0 = 2\pi/3, a_1 = 4\pi/3, a_2 = 0$, then we find that we get $\eta = (-1/3, -1/3, -1/3)$. This is to do with the fact that the Čech-de Rham isomorphism morally comes from analysis of the *total cohomology* of the Čech-de Rham double complex; hence it makes sense for us to introduce signs of $(-1)^m$ in the formulas of Theorem 4.4.6; applying this to the second formula in Theorem 4.4.6 will give us the formula for the explicit isomorphism which can be found in Proposition 9.8 of [1].

5.2. **The 2-Sphere S^2.**

Let us try the same game with the 2-sphere S^2. Firstly, we calculate the Čech cohomology of S^2. A triangulation of S^2 is a (hollow) tetrahedron:



From Proposition 2.2.12, we can thus find a good open cover $\mathfrak{U}$ of S^2 whose nerve $N(\mathfrak{U})$ is this triangulation. By Remark 2.3.9, we find the Čech complex $C^*(\mathfrak{U}, \mathbb{R})$ of the cover $\mathfrak{U}$ is

$$\begin{array}{ccccccccc} & & \mathbb{R}^4 & & \mathbb{R}^6 & & \mathbb{R}^4 & & \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & C^0(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\delta_0} & C^1(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\delta_1} & C^2(\mathfrak{U}, \mathbb{R}) & \longrightarrow & 0 \end{array}$$

As in Example 2.3.8 and Example 2.3.10, we see that $\check{H}^0\mathfrak{U}, \mathbb{R}) = \operatorname{Ker} \delta_0 \simeq \mathbb{R}$ and $\operatorname{Im} \delta_0 = \mathbb{R}^3$. Ordering the bases of $C^1(\mathfrak{U}, \mathbb{R})$ and $C^2(\mathfrak{U}, \mathbb{R})$ "lexicographically", the map

$$\delta_1 : C^1(\mathfrak{U}, \mathbb{R}) \to C^2(\mathfrak{U}, \mathbb{R})$$
$$(\eta_{01}, \eta_{02}, \eta_{03}, ..., \eta_{23}) \mapsto ((\delta\eta)_{012}, (\delta\eta)_{013}, (\delta\eta)_{023}, (\delta\eta)_{123})$$

is given by the matrix

$$\begin{pmatrix} +1 & -1 & 0 & +1 & 0 & 0 \\ +1 & 0 & -1 & 0 & +1 & 0 \\ 0 & +1 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & -1 & +1 \end{pmatrix}$$

Using some row operations, we see that this matrix has rank 3. It follows that $\operatorname{Ker}\delta_1 = \mathbb{R}^3$ and $\operatorname{Coker}\delta_1 = \mathbb{R}$. Hence, using Corollary 2.5.4, we find the Čech cohomology of S^2 is

$$\breve{H}^k(S^2, \mathbb{R}) = \breve{H}^k(\mathfrak{U}, \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

which agrees with the de Rham cohomology of S^2.

To see how the formulas of Theorem 4.4.6 work, we will need to specify a good cover $\mathfrak{U}$. Letting

$$\mathfrak{S}(\theta, \phi) := (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

we can then write $S^2 = \{\mathfrak{S}(\theta, \phi) \mid \theta \in [0, 2\pi], \phi \in [0, \pi]\}$, and we consider the open cover $\mathfrak{U} = \{U_0, ..., U_3\}$ of S^2 given by

$$\begin{aligned}
U_0 &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(0, \frac{4\pi}{3}\right), \phi \in \left[0, \frac{\pi}{2} + \varepsilon\right)\right\} \cup \mathbb{D}_\varepsilon \\
U_1 &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(\frac{2\pi}{3}, 2\pi\right), \phi \in \left[0, \frac{\pi}{2} + \varepsilon\right)\right\} \cup \mathbb{D}_\varepsilon \\
U_2 &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right), \phi \in \left[0, \frac{\pi}{2} + \varepsilon\right)\right\} \cup \mathbb{D}_\varepsilon \\
U_3 &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in [0, 2\pi], \phi \in \left(\frac{\pi}{2}, \pi\right]\right\}
\end{aligned}$$

where $\mathbb{D}_\varepsilon = \{\mathfrak{S}(\theta, \phi) \mid \theta \in [0, 2\pi], \phi \in [0, \varepsilon)\}$ and where $0 < \varepsilon < \frac{\pi}{4}$. We observe that $\mathfrak{U}$ is a good cover of S^2: the nonempty twofold intersections are

$$\begin{aligned}
U_{01} &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right), \phi \in \left[0, \frac{\pi}{2} + \varepsilon\right)\right\} \cup \mathbb{D}_\varepsilon \\
U_{02} &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(0, \frac{2\pi}{3}\right), \phi \in \left[0, \frac{\pi}{2} + \varepsilon\right)\right\} \cup \mathbb{D}_\varepsilon \\
U_{03} &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(0, \frac{4\pi}{3}\right), \phi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right)\right\} \\
U_{12} &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(\frac{4\pi}{3}, 2\pi\right), \phi \in \left[0, \frac{\pi}{2} + \varepsilon\right)\right\} \cup \mathbb{D}_\varepsilon \\
U_{13} &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(\frac{2\pi}{3}, 2\pi\right), \phi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right)\right\} \\
U_{23} &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right), \phi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right)\right\},
\end{aligned}$$

the nonempty threefold intersections are

$$\begin{aligned}
U_{012} &= \mathbb{D}_\varepsilon \\
U_{013} &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right), \phi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right)\right\} \\
U_{023} &= \left\{\mathfrak{S}(\theta, \phi) \mid \theta \in \left(0, \frac{2\pi}{3}\right), \phi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right)\right\}
\end{aligned}$$

$$U_{123} = \left\{ \mathfrak{S}(\theta, \phi) \mid \theta \in \left(\frac{2\pi}{3}, 2\pi\right), \phi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right) \right\},$$

and all further intersections are empty; the above nonempty intersections are obviously contractible. Thus, the nerve $N(\mathfrak{U})$ of $\mathfrak{U}$ is the triangulation of S^2 we had above.

Since the cover is good, we again see that $(\operatorname{Ker} d) \cap \Omega^0(S^2) = (\operatorname{Ker} \delta) \cap C^0(S^2)$ so nothing interesting happens to realise $H^0_{\mathrm{DR}} \simeq \check{H}^0(\mathfrak{U}, \mathbb{R})$. Going from Čech to de Rham, is almost exactly the same what we did for S^1. We find that

$$\eta = (\eta_{012}, \eta_{013}, \eta_{023}, \eta_{123}) = (1, 0, 0, 0)$$

is a non-trivial Čech 2-cocycle. Fixing a partition of unity subordinate to $\mathfrak{U}$, under the modified isomorphism of Theorem 4.4.6 (see Remark 5.1.1), we find that the de Rham class corresponding to $[\eta]$ has representative $\omega = (-1)^2 K(dK)^2 \eta \in \Omega^2(M)$, which one can check is a form with support in U_{012} and total integral 1.

It turns out going from de Rham to Čech is quite a bit trickier: we will just give an outline for how this should proceed. The canonical representative of a generator of $H^2_{\mathrm{DR}}(S^2)$ is $\frac{d\phi \wedge d\theta}{4\pi}$. As when we did S^1, we will use local parametrisations (trivialisations) of each of the U_i. If $r_{\alpha,\beta}$ is some origin-fixing rotation of $\mathbb{R}^3$ obtained by a rotation by α about the x-axis (anti-clockwise as we look along the negative x-direction) followed by some rotation by β about the y-axis (anti-clockwise as we look along the negative y-direction), we see that

$$\begin{aligned}
\mathfrak{S}_0(\theta, \phi) &:= r_{0,\frac{\pi}{4}} \circ \mathfrak{S}(\theta, \phi) \\
\mathfrak{S}_1(\theta, \phi) &:= r_{0,\frac{\pi}{4}} \circ \mathfrak{S}(\theta, \phi) \\
\mathfrak{S}_2(\theta, \phi) &:= r_{0,\frac{3\pi}{4}} \circ \mathfrak{S}(\theta, \phi) \\
\mathfrak{S}_3(\theta, \phi) &:= r_{\frac{\pi}{2},\pi} \circ \mathfrak{S}(\theta, \phi)
\end{aligned}$$

will be such that, for $i = 0, ..., 3$, $\mathfrak{S}_i$ restricts to give a homeomorphism of U_i onto some (contractible) open subset $V_i \subseteq \mathbb{R}^2$; we write this local parametrisation of U_i as $\mathfrak{S}_i(\theta_i, \phi_i)$ where $(\theta_i, \phi_i) \in V_i$; we write $\tau_{ij} = \left(\tau_{ij,\theta}, \tau_{ij,\phi}\right) := \mathfrak{S}_i \circ \mathfrak{S}_j^{-1}$ for the transition functions.

The vertices of $N(\mathfrak{U})$ are nonnegative integers and so are naturally ordered. For each $\sigma \in N(\mathfrak{U})$, we let σ_l denote the vertex of σ of least order; we let

$$V_\sigma = \mathfrak{S}_\sigma^{-1}(U_\sigma) := \mathfrak{S}_{\sigma_l}^{-1}(U_\sigma)$$

to get a local parametrisation $\mathfrak{S}_\sigma : V_\sigma \xrightarrow{\sim} U_\sigma$ of U_σ, which we write as $\mathfrak{S}_\sigma(\theta_\sigma, \phi_\sigma)$; we fix a differentiable retraction $\gamma_\sigma : U_\sigma \times \mathbb{R} \to U_\sigma$, and let $\mathfrak{S}_{\sigma_l}(a_\sigma, b_\sigma)$ be the constant value of γ_σ on $U_\sigma \times (-\infty, 0]$, where $(a_\sigma, b_\sigma) \in V_\sigma$. The restriction of the differentiable retractions to $U_\sigma \times [0, 1]$ are functions of the form

$$(\mathfrak{S}_\sigma(\theta_\sigma, \phi_\sigma), t) \mapsto \mathfrak{S}_\sigma \circ v^\sigma(\theta_\sigma, \phi_\sigma, t)$$

where for each $(\theta_\sigma, \phi_\sigma) \in V_\sigma$, the smooth path in V_σ

$$v^\sigma(\theta_\sigma, \phi_\sigma, -) : [0, 1] \to V_\sigma; t \mapsto (v_1^\sigma(\theta_\sigma, \phi_\sigma, t), v_2^\sigma(\theta_\sigma, \phi_\sigma, t))$$

has $v^\sigma(\theta_\sigma, \phi_\sigma, 0) = (\theta_\sigma, \phi_\sigma)$ and $v^\sigma(\theta_\sigma, \phi_\sigma, 1) = (a_\sigma, b_\sigma)$. By comparison with the computations in Section 8, we see that for $i = 0, 1, ..., 3$, we have

$$
\begin{aligned}
I_i r\left(\frac{d\phi\wedge d\theta}{4\pi}\right)_i &= \frac{1}{4\pi}\left(\int_0^1 \frac{\partial v_2^i}{\partial t_1}\frac{\partial v_1^i}{\partial \phi_i} - \frac{\partial v_2^i}{\partial \phi_i}\frac{\partial v_1^i}{\partial t_1}\,\mathrm{d}t_1\right)d\phi_i \\
&\quad + \frac{1}{4\pi}\left(\int_0^1 \frac{\partial v_2^i}{\partial t_1}\frac{\partial v_1^i}{\partial \theta_i} - \frac{\partial v_2^i}{\partial \theta_i}\frac{\partial v_1^i}{\partial t_1}\,\mathrm{d}t_1\right)d\theta_i
\end{aligned}
$$

For $i = 0, ..., 3$ we let

$$
\begin{aligned}
f_i(\theta_i,\phi_i) &= \frac{1}{4\pi}\int_0^1 \frac{\partial v_2^i}{\partial t_1}\frac{\partial v_1^i}{\partial \phi_i} - \frac{\partial v_2^i}{\partial \phi_i}\frac{\partial v_1^i}{\partial t_1}\,\mathrm{d}t_1 \\
g_i(\theta_i,\phi_i) &= \frac{1}{4\pi}\int_0^1 \frac{\partial v_2^i}{\partial t_1}\frac{\partial v_1^i}{\partial \theta_i} - \frac{\partial v_2^i}{\partial \theta_i}\frac{\partial v_1^i}{\partial t_1}\,\mathrm{d}t_1
\end{aligned}
$$

so that

$$
\begin{aligned}
(\delta I r)\left(\frac{d\phi\wedge d\theta}{4\pi}\right)_{ij} &= f_j d\phi_j + g_j d\theta_j - f_i d\phi_i - g_i d\theta_i \\
&= \left((f_j\circ\tau_{ji})\left(\frac{\partial \tau_{ji,\phi}}{\partial \phi_i}\right) + (g_j\circ\tau_{ji})\left(\frac{\partial \tau_{ji,\theta}}{\partial \phi_i}\right) - f_i\right)d\phi_i \\
&\quad + \left((f_j\circ\tau_{ji})\left(\frac{\partial \tau_{ji,\phi}}{\partial \theta_i}\right) + (g_j\circ\tau_{ji})\left(\frac{\partial \tau_{ji,\theta}}{\partial \theta_i}\right) - g_i\right)d\theta_i
\end{aligned}
$$

Letting

$$
\begin{aligned}
G_{ij} &:= (f_j\circ\tau_{ji})\left(\frac{\partial \tau_{ji,\phi}}{\partial \phi_i}\right) + (g_j\circ\tau_{ji})\left(\frac{\partial \tau_{ji,\theta}}{\partial \phi_i}\right) - f_i \\
F_{ij} &:= (f_j\circ\tau_{ji})\left(\frac{\partial \tau_{ji,\phi}}{\partial \theta_i}\right) + (g_j\circ\tau_{ji})\left(\frac{\partial \tau_{ji,\theta}}{\partial \theta_i}\right) - g_i
\end{aligned}
$$

for $i < j$, we have

$$
(I\delta I r)\left(\frac{d\phi\wedge d\theta}{4\pi}\right)_{ij} = \int_0^1 G_{ij}\frac{\partial v_2^{ij}}{\partial t_2} + F_{ij}\frac{\partial v_1^{ij}}{\partial t_2}\,\mathrm{d}t_2
$$

so that for $i < j < k$, we have

$$
\begin{aligned}
&(\delta I\delta I r)\left(\frac{d\phi\wedge d\theta}{4\pi}\right)_{ijk} \\
&= (I\delta I r)\left(\frac{d\phi\wedge d\theta}{4\pi}\right)_{jk} - (I\delta I r)\left(\frac{d\phi\wedge d\theta}{4\pi}\right)_{ik} + (I\delta I r)\left(\frac{d\phi\wedge d\theta}{4\pi}\right)_{ij} \\
&= \int_0^1 \left(G_{jk}\frac{\partial v_2^{jk}}{\partial t_2} + F_{jk}\frac{\partial v_1^{jk}}{\partial t_2} - G_{ik}\frac{\partial v_2^{ik}}{\partial t_2} - F_{ik}\frac{\partial v_1^{ik}}{\partial t_2} + G_{ij}\frac{\partial v_2^{ij}}{\partial t_2} + F_{ij}\frac{\partial v_1^{ij}}{\partial t_2}\right)\mathrm{d}t_2
\end{aligned}
$$

We omit the verification that this is in fact constant.

6. Some Closing Thoughts

In this section, I give my interpretation of how the Čech-de Rham isomorphism works, which is that it is essentially just homotopy invariance of de Rham cohomology. For simplicity, we will assume further that M is oriented. As in the proof of Theorem 2.5.3, since the set of good covers of a manifold M is cofinal, to understand why the Čech and de Rham cohomologies of M are isomorphic, you only really need to understand the isomorphism for good covers of M. Now, recall from Remark 2.2.10 that the nerve $N(\mathfrak{U})$ of a good cover $\mathfrak{U}$ of a manifold M is homotopy equivalent to M. Using the interpretation of Čech cochains stated in Remark 2.3.2 (and after a suitable embedding of the nerve $N(\mathfrak{U})$ into M), we may think of the formulas of Theorem 4.4.6 as: how to "squeeze" a differential m-form on M into locally constant functions on each of the (oriented) m-simplices of $N(\mathfrak{U})$; and conversely how to "smooth-out" a Čech m-cochain into a differential m-form on M.

For the "squeezing" operation, we think of a differential m-form ω on a manifold M as a sum of weighted "m-dimensional" volume elements. We then think of the operator I as first "re-orienting" ω (by pulling back the form by the differentiable retraction) so that one of its dimensions is the new "fake" dimension dt, and then squeezing it along the various paths $v^\sigma(\theta_\sigma, \phi_\sigma, 1)$ (by integrating with respect to t). The difference operator δ combines the various integrals along the boundaries of a simplex via an oriented sum. Noting that this is reminiscent of Stokes' Theorem (Theorem 3.3.6), we can think of (δI) as integrating along 1-simplices, and hence $(\delta I)^m$ as integrating on m-simplices.

The mindset that Čech-de Rham is "merely homotopy invariance of de Rham cohomology" is supported by an alternative proof of the Čech-de Rham isomorphism using sheaf theory (see, for example, Appendix A of [2] or p.44 of [11]) where the Poincaré Lemma now asserts that the de Rham complex on a manifold M is a sheaf resolution of the constant sheaf $\mathbb{R}$ on M.

7. Further Directions

In this section I briefly discuss some potential directions for further study.

The obvious one is of course to study the proof of the Čech-de Rham isomorphism from the perspective of sheaf theory; the theory of sheaves being of interest in their own right. More generally, one can look at cohomology of sheaves on a topological space X, which often coincides with the Čech cohomology of X when X is a reasonable space (p.293 of [2]).

Another potential direction (which was suggested to me by my tutor Prof. Kobi Kremnizer) is to study a "combinatorial Poincaré duality". For a manifold M, there is a duality pairing between the singular homology of M and the de Rham cohomology of M called Poincaré duality. Since the singular and simplicial homology of M are isomorphic, and the Čech and de Rham cohomology of M are isomorphic, there is thus a duality paiting between the simplicial homology of M and the Čech cohomology of M. The question is, can one realise this duality pairing directly without passing through the singular-simplicial and Čech-de Rham isomorphisms?

8. Appendix: Finishing Off the Proof of Lemma 4.3.1

Proof. Let $q > 0$. If $\omega \in \Omega^q(U)$, then for $x \in U \cap U_\beta$, ω can be written in local coordinates as $\omega = \sum_{(i)} f_{(i)} dx_{i_1} \cdots dx_{i_q}$ where this sum is over multi-indices (i) of length q. Since I and d are both linear, it suffices to verify the claim when

$$\omega = f dx_1 \cdots dx_q$$

is a monomial where, without loss of generality, we have relabelled the x_i's so that $(i_1, \ldots, i_q) = (1, \ldots, q)$.

Letting $\gamma_i = x_i \circ \gamma$ be the i^{th} component of γ, since d commutes with pullbacks (Remark 3.2.2), we have for $x \in U \cap U_\alpha$

$$\gamma^* \omega = \gamma^* (f dx_1 \cdots dx_q) = (f \circ \gamma) d\gamma_1 \cdots d\gamma_q$$

Temporarily writing $t = x_0$, we have

$$\begin{aligned}
d\gamma_1 \cdots d\gamma_q &= \bigwedge_{i=1}^{q} \left(\sum_{k=0}^{n} \frac{\partial \gamma_i}{\partial x_k} dx_k \right) \\
&= \sum_{k_1, \ldots, k_q = 0}^{n} \left(\frac{\partial \gamma_1}{\partial x_{k_1}} \cdots \frac{\partial \gamma_q}{\partial x_{k_q}} \right) dx_{k_1} \cdots dx_{k_q} \\
&= \sum_{i=0}^{q} \left[\sum_{\substack{k_i = 0 \\ \text{other } k_j > 0}} \left(\frac{\partial \gamma_1}{\partial x_{k_1}} \cdots \frac{\partial \gamma_q}{\partial x_{k_q}} \right) dx_{k_1} \cdots dx_{i-1} dt dx_{i+1} \cdots dx_{k_q} \right] \\
&\quad + \sum_{k_1, \ldots, k_q > 0} \left(\frac{\partial \gamma_1}{\partial x_{k_1}} \cdots \frac{\partial \gamma_q}{\partial x_{k_q}} \right) dx_{k_1} \cdots dx_{k_q} \\
&= \sum_{i=0}^{q} \left[\sum_{\substack{k_i = 0 \\ \text{other } k_j > 0}} (-1)^{i-1} \left(\frac{\partial \gamma_1}{\partial x_{k_1}} \cdots \frac{\partial \gamma_q}{\partial x_{k_q}} \right) dt dx_{k_1} \cdots dx_{i-1} dx_{i+1} \cdots dx_{k_q} \right] \\
&\quad + \sum_{k_1, \ldots, k_q > 0} \left(\frac{\partial \gamma_1}{\partial x_{k_1}} \cdots \frac{\partial \gamma_q}{\partial x_{k_q}} \right) dx_{k_1} \cdots dx_{k_q}
\end{aligned}$$

which allows us to write $\gamma^* \omega$ in the form of Equation 1, by setting

$$\begin{aligned}
f_{(i)}(x, t) &= (f \circ \gamma(x, t)) \cdot \left(\frac{\partial \gamma_1}{\partial x_{k_1}} \cdots \frac{\partial \gamma_q}{\partial x_{k_q}} \right) \\
g_{(j)} &= (f \circ \gamma(x, t)) \cdot (-1)^{i-1} \left(\frac{\partial \gamma_1}{\partial x_{k_1}} \cdots \frac{\partial \gamma_{i-1}}{\partial x_{k_{i-1}}} \frac{\partial \gamma_i}{\partial t} \frac{\partial \gamma_{i+1}}{\partial x_{k_{i+1}}} \cdots \frac{\partial \gamma_q}{\partial x_{k_q}} \right)
\end{aligned}$$

for multi-indices $(i) = (k_1, \ldots, k_q)$ and $(j) = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_q)$. We then have

$$\begin{aligned}
&f_{(i)}(x, 1) - f_{(i)}(x, 0) \\
&= f(x) \cdot \left(\frac{\partial \gamma_1}{\partial x_{k_1}} \cdots \frac{\partial \gamma_q}{\partial x_{k_q}} \right)(x, 1) - f(a) \cdot 0 \qquad \text{since } \gamma \equiv a \text{ on } U \times (-\infty, 0] \\
&= \begin{cases} f(x) & \text{if } (1, \ldots, q) = (k_1, \ldots, k_q) \\ 0 & \text{otherwise.} \end{cases} \qquad \text{since } \gamma(x, 1) \equiv x
\end{aligned}$$

We can now verify

$$Id\omega + dI\omega = \sum_{(i)} \left[f_{(i)}(x,1) - f_{(i)}(x,0) \right] dx_{i_1} \cdots dx_{i_q} = f(x) dx_1 \cdots dx_q = \omega$$

completing the proof of Lemma 4.3.1. □

9. Appendix: Barycentric Subdivision

This section can be treated as a continuation of the line of thought started in Proposition 2.2.12. In the definition of Čech cohomology, we repeatedly refine open covers $\mathfrak{U}$ of a topological space X (by taking a direct limit), giving us increasingly better combinatorial appromxations of X via the nerve $N(\mathfrak{U})$. This section gives us a converse of this process: we start with a triangulation of a space X which we repeatedly refine to generate a cofinal set of good covers of X. This give us an alternative proof of Corollary 2.2.16 (with the further assumption that M is compact, though I am not sure this condition is strictly necessary).

A standard way to subdivide a simplex Δ into smaller simplices is *barycentric subdivision*; we denote by $\operatorname{sd} \Delta$ the barycentric subdivision of Δ. If K is a simplicial complex, then the barycentric subdivision of K is the simplicial complex $\operatorname{sd} K$ obtained by applying barycentric subdivision to each of the simplices of K; it is a *subdivision* of K; that is, the geometric realisation of every simplex of $\operatorname{sd} K$ is contained in the geometric realisation of a simplex of K, and the geometric realisation of each simplex of K is a union of the geometric realisations of finitely many simplices of $\operatorname{sd} K$. For $k > 0$, we denote by $\operatorname{sd}^k K$ the k-th barycentric subdivision of K, that is, the simplicial complex obtained by applying barycentric subdivision to K k-times; there is a natural inclusion $\left|\operatorname{sd}^k K\right| \hookrightarrow |K|$. Further details regarding barycentric subdivision can be found in §15 of [12].

Lemma 9.1. *Let M be a compact n-dimensional manifold, and fix a triangulation K of M (furnished by Theorem 2.2.13) and a homeomorphism $\varphi : |K| \xrightarrow{\sim} M$. For $k > 0$, let $\mathfrak{U}^{(k)}$ be the good open cover of M arising from $\operatorname{sd}^k K$ by taking open stars of its vertices (Proposition 2.2.12). Then the set $\left\{\mathfrak{U}^{(k)} \mid k > 0\right\}$ is cofinal in the set of open covers of M ordered under refinement.*

Proof. Let $\mathfrak{V}$ be an open cover of M. By the Whitney Embedding Theorem (Theorem 6.15 of [6]), M admits a smooth embedding into $\mathbb{R}^{2n+1}$; this induces a metric $\mathfrak{d}$ on M and thus makes M into a metric space. Since M is compact, the Lebesgue number lemma (Lemma 27.5 of [13]) shows that there is $r > 0$ such that whenever $S \subseteq M$ is a bounded subset of M with

$$\operatorname{diam}(S) := \sup\{\mathfrak{d}(s_1, s_2) \mid s_1, s_2 \in S\} < r$$

then there is an open set $V \in \mathfrak{V}$ such that $S \subseteq V$.

The metric $\mathfrak{d}$ on M induces a metric on $|K|$ through φ, and, since M is compact, K must be a finite simplicial complex. Hence Theorem 15.4 of [12] shows that there is $k > 0$ such that for each simplex σ of $\operatorname{sd}^k K$, we have

$$\operatorname{diam}(\varphi(\Delta_\sigma)) < \frac{r}{2}.$$

Thus, for any vertex v of $\operatorname{sd}^k K$, we see that

$$\operatorname{diam}(\varphi(\operatorname{st} v)) < r$$

and hence $\varphi(\operatorname{st} v) \subseteq V$ for some open set $V \in \mathfrak{V}$. Thus, every open set in $\mathfrak{U}^{(k)}$ is contained in some open set in $\mathfrak{V}$; that is, $\mathfrak{U}^{(k)}$ is a refinement of $\mathfrak{V}$. □

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